Connectedness, compactness implies continuity in Euclidean n-space R^n

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Abstract

We know the popular result in multivariable analysis that "A continuous multivariable function maps connected set to connected set and compact set to compact set." I have proved the converse is true in R i.e. "A real function which maps connected set to connected set and compact set to compact set is continuous." This article generalizes the result to Euclidean n-space R^n.

Theorem:

A function f: $R^m \to R^k$ which maps connected set to connected set and compact set to compact set is continuous.

Proof:

Let f: $R^m \to R^k$ maps connected set to connected set and compact set to compact set.

We can write $f(x) = (f_1(x), f_2(x), f_3(x) \ldots f_k(x))$.

Since f maps connected set to connected set and compact set to compact set.

Hence f_i maps connected set to connected set and compact set to compact set for each i.

(As "f_i is composite function of f and projection function $R^k \to R$" and projection function is continuous function)

We know a result in multivariable analysis that

$f : R^m \to R^k$ is continuous if and only if $f_i : R^m \to R$ is continuous for $1 \leq i \leq k$.

To prove that f is continuous on R^m.

Therefore, it sufficient to prove that f_i is continuous on R^m for each i.

Suppose f_i is not continuous at a point x in R^m.

$\therefore \exists \in > 0$, for a given $\delta > 0 \; \exists \; z(\delta) = z \in R^m$ such that

$$\|z - x\| < \delta \Rightarrow |f_i(z) - f_i(x)| > \in > 0$$

For $\delta = 1, \exists \; x_1 \in R^m$, $\|x_1 - x\| < \delta \Rightarrow |f_i(x_1) - f_i(x)| > \in > 0$

For each $n \in N - \{1\}$.

Let $\delta_n = \min\{\frac{1}{3^n}, \|x_{n-1}\text{-}x\|\}$.

Then $\exists$ $x_n \in R^m$, $\|x_n\text{-}x\| < \delta_n \Rightarrow |f_i(x_n) - f_i(x)| > \in > 0$.

$\therefore 0 < \|x_n\text{-}x\| < \delta_n \leq \|x_{n-1}\text{-}x\|$ and $\|x_n\text{-}x\| < \delta_n \leq \frac{1}{3^n}$. Also $x_n \in B[x, \delta_n]$.

Thus $\exists$ a sequence of distinct point $x_n \to x$ and $f_i(x_n) \nrightarrow f_i(x)$.

Consider a nested sequence of closed, bounded balls

$B[x, \delta_1] \supset B[x, \delta_2] \supset B[x, \delta_3] \supset \cdots \supset B[x, \delta_n] \supset \cdots$

Since any closed and bounded ball in R^m is connected and compact set.

$\therefore B[x, \delta_1], B[x, \delta_2], B[x, \delta_3], \ldots, B[x, \delta_n], \ldots$ is a nested sequence of connected and compact sets.

Hence $f_i(B[x, \delta_1]), f_i(B[x, \delta_2]), f_i(B[x, \delta_3]), \ldots, f_i(B[x, \delta_n]), \ldots$ is a nested sequence of connected and compact sets in R. {$\because f_i$ maps connected set to connected set and compact set to compact set}.

Also $f_i(B[x, \delta_1]), f_i(B[x, \delta_2]), f_i(B[x, \delta_3]), \ldots, f_i(B[x, \delta_n]), \ldots$ is a nested sequence of closed and bounded intervals in R. {$\because$ in R, a connected set is an interval and a compact set is closed and bounded set}.

Since arbitrary intersection of nested sequence of closed and bounded balls in R^m is a non empty closed and bounded ball. {By intersection property of connected and compact sets}

Therefore, $\bigcap_{n \in N} B[x, \delta_n]$ is closed and bounded ball and $\bigcap_{n \in N} f_i(B[x, \delta_n])$ is closed and bounded interval.

$\because \delta_n \to 0$.

$$\therefore \bigcap_{n \in N} B[x, \delta_n] = \{x\}.$$

Now, $f_i(x_n) \in f_i(B[x, \delta_1])$ $\forall\, n \in N$ and $f_i(B[x, \delta_1])$ is closed and bounded set.

Hence there exists a convergent subsequence of a sequence $f_i(x_n)$ which converges to $y \in f_i(B[x, \delta_1])$ and $y \neq f_i(x)$.

Clearly, $y \in \bigcap_{n \in N} f_i(B[x, \delta_n])$.

Thus $\bigcap_{n \in N} f_i(B[x, \delta_n])$ is a closed and bounded interval containing two distinct points y and $f_i(x)$.

Let $\bigcap_{n\in N} f_i(B[x,\delta_n]) = [\alpha,\beta]$ where for some distinct $\alpha,\beta \in R$.

Now, we can find a sequence of distinct points $y_n \in [\alpha,\beta]$ and $y_n \neq y$ which converges to y and get a corresponding sequence of distinct points z_n such that for each $n \in N$, $z_n \neq x, z_n \in B[x,\delta_n]$ and $y_n = f_i(z_n) \in [\alpha,\beta]$.

$\because \delta_n \to 0$ and $z_n \in B[x,\delta_n]$.

$\therefore z_n \to x$.

$\therefore \{z_n : n \in N\} \cup \{x\}$ is a closed and bounded set and hence a compact set.

Its image set $\{f_i(z_n) : n \in N\} \cup \{f_i(x)\}$ is compact set $\{\because f_i$ takes compact set to compact set.$\}$

But $y_n = f_i(z_n) \neq y$ and $y_n \to y$.

Which is contradiction to the statement that "$\{f_i(z_n) : n \in N\} \cup \{f_i(x)\}$ is compact set and hence a closed set". $\{\because$ every convergent sequence in closed set converges to a point in it.$\}$

Our assumption that "*f_i is not continuous at x*" is wrong.

Hence f_i is continuous at a point x in R^m and on R^m.

Thus f_i is continuous on R^m for each i which implies f is continuous on R^m.

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