Remarks on ideal in Distributive Q-lattices

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Abstract

In this paper, we prove if J be an ideal of distributive q-lattice A then for any prime ideal P containing J, J(P) is an ideal of A such that J⊆ J(P) ⊆ P also if P be a prime ideal containing an ideal J of distributive q-lattice A and Q be a prime ideal such that J ⊆ Q ⊆ P then J(P) ⊆ Q. We define a relation Θ_J on A and prove is congruence relation on A under condition and for f: $A_1 \rightarrow A_2$ be a homomorphism, we prove if J is an ideal of A_2 then $f^{-1}(J)$ is an ideal of A_1 and similar for filter.

KEYWORDS: Distributive q-lattice, Ideal , Congruence equation.

I. INTRODUCTION

Ivan Chajda [2] introduced the concept of a q-lattice and defined distributive q-lattice. After that G. C. Rao, P. Sundarayya, S. Kalesha vali, and Ravi Kumar Bandaru [1] , they defined ideals of a distributive q-lattice. They proved if A be a distributive q-lattice then I (A), the set of all ideals of A is a lattice under set inclusion. They give some equivalent conditions of a distributive q-lattice to become a distributive lattice in terms of ideals. In paper Filter and Annihilator in Distributive q-lattices, A. D. Lokhande, Ashok S Kulkarni [4] , defined Filter in a distributive q-lattice and proved if A be a distributive q-lattice then F(A), the set of all filters of A is a lattice under set inclusion. G. C. Rao and M. Sambasiva Rao [5] defined ‘ annihilator ’ in Almost Distributive Lattice (ADL_S) and derived some properties, In paper [4] we defined annihilator in distributive q-lattice A and proved for any ideal I of distributive q-lattice A and a∈ A , the annihilator (a:I) is an ideal of A and derived some properties. In this paper we define J(P) and prove if J be an ideal of distributive q-lattice A then for any prime ideal P containing J, J(P) is an ideal of A such that J⊆J(P) ⊆P also if P be a prime ideal containing an ideal J of distributive q-lattice A and Q be a prime ideal such that J ⊆ Q ⊆ P then J(P) ⊆ Q. we show (a] = { a ∧ x / a ∈ {a} , x ∈ A } is an ideal of A. We define a relation Θ_J on A and prove is congruence relation on A under some condition. and for f: $A_1 \rightarrow A_2$ be a homomorphism we prove If J is an ideal of A_2 then $f^{-1}(J)$ } is an ideal of A_1 and similar for filter

II PRELIMINARIES

Some of the following definitions and results are taken from [1] and [4]

Definition II.1:[1]. An algebra $(A, \vee, \wedge)$ whose binary operations $\vee, \wedge$ satisfy the following is called a q-lattice.

(i) $a \vee b = b \vee a$; $a \wedge b = b \wedge a$ (commutativity)
(ii) $a \vee (b \vee c) = (a \vee b) \vee c$; $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (associatativity)
(iii) $a \vee (a \wedge b) = a \vee a$; $a \wedge (a \vee b) = a \wedge a$ (weak- absorption)
(iv) $a \vee b = a \vee (b \vee b)$; $a \wedge b = a \wedge (b \wedge b)$ (weak- idempotence)
(v) $a \vee a = a \wedge a$ (equalization)

Definition II.2:[1]. A q-lattice $(A, \vee, \wedge)$ is distributive if it satisfies the identity

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in A$

Lemma II.I :[1]. Let A be a distributive q-lattice then the following identity hold

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in A$.

Definition II.3:[1]. Ideal of a distributive q-lattice:

A nonempty subset I of a distributive q-lattice A is called an ideal of A if

i) $x, y \in I \Rightarrow x \vee y \in I$
ii) $x \in I$ and $a \in A \Rightarrow a \wedge x \in I$

Definition II.4:[4]. Filter of a distributive q- lattice :

A nonempty subset F of a distributive q-lattice A is called a filter of A, if.

i) $x, y \in F \Rightarrow x \wedge y \in F$
ii) $x \in F$ and $a \in A \Rightarrow a \vee x \in F$

Definition II.5 :[4].

Let A be distributive q-lattice and I is an ideal of A then for any non empty subset P of A define annihilator

$P^* = (P: I) = \{x \in A / x \wedge a \in I, \text{ for all } a \in P\}$

If $P = \{a\}$ then we write $(\{a\}: J)$ as the annihilator $(a: J)$ of $a \in P$

Therefore $(a: I)=\{x \in A / x \wedge a \in I, \text{ for all } a \in \{a\}\}$

III. Prime ideal J(P):

Definition: III.1:

A proper ideal I of a distributive q-lattice A is called a prime ideal if for any x ,

$y \in A$, $x \wedge y \in I$ implies $x \in I$ or $y \in I$

Definition: III.2:

A prime ideal P of A is called a minimal prime ideal belonging to an ideal I, if (i). $I \subseteq P$ and (ii).there is no prime ideal Q such that $I \subseteq Q \subset P$.

Definition: III.3:

Let P be a prime ideal containing an ideal J of distributive q-lattice A then define

$$J(P) = \{ x \in A / x \wedge t \in J \text{ for some } t \notin P\}$$

$$= \{ x \in A / t \wedge x \in J \text{ for some } t \notin P\}$$

Lemma:III.1

Let J be an ideal of distributive q-lattice A then for any prime ideal P containing J, J(P) is an ideal of A such that $J \subseteq J(P) \subseteq P$

Proof : Let P be a prime ideal containing an ideal J in A

Let $x \in J$, then as J is an ideal therefore $a \wedge x \in J$ for any $a \in A$

i.e. $a \wedge x = x \wedge a \in J$ for some $a \notin P$ also

this implies $x \in J(P)$ therefore $J \subseteq J(P)$

Now to show J(P) is an ideal:

Let $a, b \in J(P)$ implies $x \wedge a = a \wedge x \in J$ and $y \wedge b = b \wedge y \in J$ for some $x \notin P$ and $y \notin P$

Consider $(x \wedge y) \wedge (a \vee b) = ((x \wedge y) \wedge a) \vee ((x \wedge y) \wedge b)$

Now as P is prime ideal and $x \notin P$, $y \notin P$ implies $x \wedge y \notin P$

$= (x \wedge (y \wedge a)) \vee (x \wedge (y \wedge b))$

$= (x \wedge (a \wedge y)) \vee (x \wedge (y \wedge b))$

$= ((x \wedge a) \wedge y) \vee (x \wedge (y \wedge b)$

As $x \wedge a \in J$, $y \in A$

Implies $(x \wedge a) \wedge y \in J$

And $y \wedge b \in J$, $x \in A$

Implies $x \wedge (y \wedge b) \in J$

And as J is an ideal

Implies $((x \wedge a) \wedge y) \vee (x \wedge (y \wedge b)) \in J$

Implies $(x \wedge y) \wedge (a \vee b) \in J$ where $x \wedge y \notin P$

Implies $a \vee b \in J(P)$

Now let $a \in J(P)$ and $r \in A$

Then $x \wedge a \in J$ for some $x \notin P$

Now to show $r \wedge a \in J(P)$

Consider $x \wedge (r \wedge a) = x \wedge (a \wedge r)$

$= (x \wedge a) \wedge r$

As $x \wedge a \in J$ and $r \in A$ and J is an ideal so $(x \wedge a) \wedge r \in J$

Implies $x \wedge (r \wedge a) \in J$ and $x \notin P$

Implies $r \wedge a \in J(P)$

Therefore J(P) is an ideal of A such that $J \subseteq J(P)$

Now let $a \in J(P)$

Then $x \wedge a \in J$ for some $x \notin P$

Implies $x \wedge a \in P$ since $J \subseteq P$

And since P is prime ideal and $x \notin P$

Implies $a \in P$

Implies $J(P) \subseteq P$

Theorem:III.1:

Let P be a prime ideal containing an ideal J of distributive q-lattice A and Q be a prime ideal such that $J \subseteq Q \subseteq P$ then $J(P) \subseteq Q$

Proof: We know $J(P) = \{ x \in A / x \wedge t \in J$, for some $t \notin P\}$

$= \{ x \in A / t \wedge x \in J$, for some $t \notin P\}$

By previous lemma $J \subseteq J(P) \subseteq P$

Let Q be a prime ideal such that $J \subseteq Q \subseteq P$

Now let $x \in J(P)$

Implies $t \wedge x \in J \subseteq Q$ for some $t \notin P$

Suppose $x \notin Q$

Since Q is prime, we get $t \in Q \subseteq P$

Which is contradiction to $t \notin P$

Thus $x \in Q$

Implies $J(P) \subseteq Q$

Theorem:III.2:

Let A be distributive q-lattice then for any ideal J of A , then

Following statements are equivalent

(a) for any a, b ∈ A , a v b ∈ J implies

$(a:J) \cup (b:J) = A$

(b) For any a ,b ∈ A , $(a \vee b: J) \subseteq (a : J) \cup (b :J)$

Proof:

(a) implies (b)

Assume for any a, b ∈ A , a v b ∈ J implies

$(a:J) \cup (b:J) = A$

Let $w \in (a \vee b :J)$ then $w \wedge (a \vee b) \in J$

That is $(w\wedge a) \vee (w\wedge b) \in J$ therefore from condition (a)

$(w \wedge a : J) \cup (w \wedge b : J) = A$

Now $w \in A$ implies $w \in (w\wedge a : J)$ or $w \in (w\wedge b : J)$

Implies $w \wedge (w \wedge a) \in J$ or $w \wedge (w \wedge b) \in J$

Implies $(w \wedge a) \in J$ or $(w \wedge b) \in J$

Implies $w \in (a : J)$ or $w \in (b : J)$

Implies $w \in (a : J) \cup (b : J)$

Implies $(a \vee b: J) \subseteq (a : J) \cup (b :J)$

(b) implies (a)

for any a, b ∈ A , let a v b ∈ J

To prove that $(a:J) \cup (b:J) = A$

Clearly $(a:J) \cup (b:J) \subseteq A$

Now , Let $x \in A$, $a \vee b \in J$ and J is an ideal

Implies $x \wedge (a \vee b) \in J$

Implies x ∈ (a ∧ b : J) therefore from (b)

Implies x ∈ (a:J) ∪ (b:J)

Therefore A ⊆ (a:J) ∪ (b:J)

Hence (a:J) ∪ (b:J) =A

Theorem: III.3

Let A be a distributive q-lattice then (a] = { a ∧ x / a ∈ {a} , x ∈ A } is an ideal of A

Proof : Let a ∧ x , a ∧ y ∈ (a]

Implies (a ∧ x) v (a ∧ y) = a ∧ (x v y) ∈ (a] since x ,y ∈ A then x v y ∈ A.

And for a ∧ x ∈ (a] , r ∈ A then

(a ∧ x) ∧ r = a ∧ (x ∧ r) ∈ (a] since x ,r ∈ A then x ∧ r ∈ A.

Therefore (a] is an ideal of distributive q- lattice A

Lemma :III.2

Let P be a nonempty set and J an ideal of distributive q-lattice A, then for any a∈ A, we have the following

i) $\bigcap_{x \in P}$ (x : J) = (P : J)

ii) (a : J) is the least upper bound for all (a : (t]) with t ∈ J.

Proof: i) suppose t ∈ (P :J)

Implies t ∧ x ∈ J for all x ∈ P

Hence t ∈ (x :J) for all x ∈ P

Implies t ∈ $\bigcap_{x \in P}$ therefore (P:J) ⊆ $\bigcap_{x \in P}$ (x : J)

Conversely let t ∈ $\bigcap_{x \in P}$ (x : J)

Implies t ∈ (x :J) for all x ∈ P

Implies t ∧ x ∈ J for all x ∈ P, hence t ∈ (P:J)

Therefore $\bigcap_{x \in P}$ (x : J) ⊆ (P:J)

Hence $\bigcap_{x \in P}$ (x : J) = (P : J)

ii) Let $x \in (a: (t])$ implies $x \wedge a \in (t]$ that is $x \wedge a = t \wedge x$ for $x \in A$

if $t \in J$ then $t \wedge x \in J$ therefore $x \wedge a \in J$ so $x \in (a : J)$

so for any $t \in J$, $(a : (t]) \subseteq (a : J)$

Thus $(a : J)$ is an upper bound for all $(a: (t])$ with $t \in J$.

Let K be any ideal in A such that $(a : (t]) \subseteq K$ for any $t \in J$.

Suppose $x \in (a : J)$ then $x \wedge a \in J$. Hence $(a : (x \wedge a]) \subseteq K$

Since $x \wedge a \in (x \wedge a]$ implies $x \in (a : (x \wedge a]) \subseteq K$ therefore $(a : J) \subseteq K$

Therefore $(a : J)$ is the least upper bound for all $(a : (t])$ with $t \in J$.

Theorem:III.4:

Let A be distributive q-lattice. Define a relation $(a, b) \in \Theta_J$ if and only if $(a :J) = (b:J)$ then Θ_J is a equivalence relation on A and J is congruence relation on A under condition $(a \vee c: J) \subseteq (a :J) \cap (c : J)$

Proof : i)As $(a:J) = (a:J)$ implies $(a , a) \in \Theta_J$ for all $a \in A$

Therefore Θ_J is reflexive

ii) If $(a, b) \in \Theta_J$ implies $(a:J) = (b:J)$ implies $(b:J) = (a:J)$

Implies $(b, a) \in \Theta_J$ therefore relation Θ_J is symmetric

iii) If $(a, b) \in \Theta_J$, $(b, c) \in \Theta_J$

Implies $(a:J) = (b:J)$, $(b:J) = (c:J)$

Implies $(a:J) = (b:J) = (c:J)$

Implies $(a:J) = (c:J)$

Implies $(a ,c) \in \Theta_J$ therefore relation Θ_J is transitive

Therefore relation Θ_J is an equivalence relation on A

Now let (a, b) , $(c ,d) \in \Theta_J$

Implies $(a:J) = (b:J)$ and $(c:J) = (d :J)$

Now, let $x \in (a \wedge c :J)$

$\Leftrightarrow x \wedge (a \wedge c) \in J$

$\Leftrightarrow (x \wedge a) \wedge c \in J$

$\Leftrightarrow x \wedge a \in (c : J)$

$\Leftrightarrow x \wedge a \in (d : J)$

$\Leftrightarrow (x \wedge a) \wedge d \in J$

$\Leftrightarrow x \wedge (a \wedge d) \in J$

$\Leftrightarrow x \wedge (d \wedge a) \in J$

$\Leftrightarrow (x \wedge d) \wedge a \in J$

$\Leftrightarrow x \wedge d \in (a : J)$

$\Leftrightarrow x \wedge d \in (b : J)$

$\Leftrightarrow (x \wedge d) \wedge b \in J$

$\Leftrightarrow x \wedge (d \wedge b) \in J$

$\Leftrightarrow x \wedge (b \wedge d) \in J$

$\Leftrightarrow x \in (b \wedge d : J)$

Hence $(a \wedge c : J) = (b \wedge d : J)$

Therefore $(a \wedge c , b \wedge d) \in \Theta_J$

Now to show $(a \vee c , b \vee d) \in \Theta_J$

Let $x \in (a : J) \cap (c : J)$

Implies $x \in (a : J)$ and $x \in (c : J)$

Implies $x \wedge a \in J$ and $x \wedge c \in J$ and as J is an ideal

Implies $(x \wedge a) \vee (x \wedge c) \in J$

Implies $x \wedge (a \vee c) \in J$

Implies $x \in (a \vee c : J)$

Hence $(a : J) \cap (c : J) \subseteq (a \vee c : J)$ and from given condition

$(a \vee c : J) \subseteq (a : j) \cap (c : J)$

Therefore $(a \vee c : J) = (a : j) \cap (c : J)$

Implies $(a \vee c : J) = (b : j) \cap (d : J)$

Implies $(a \vee c : J) = (b \vee d : j)$

Hence $(a \vee c , b \vee d) \in \Theta_J$

Theorem:III.5:

Let A_1 and A_2 be two distributive q-lattices and f: $A_1 \rightarrow A_2$ be a homomorphism then

1) If J is an ideal of A_2 then $f^{-1}(J) = \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$ is an ideal of A_1

2) If J is filter of A_2 then $f^{-1}(J) = \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$ is filter of A_1

Proof: Let f: $A_1 \rightarrow A_2$ be a homomorphism

1) Let J is an ideal of A_2 and $f^{-1} = \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$

Let $x , y \in f^{-1}(J)$ then $f(x) , f(y) \in J$

As J is an ideal , $f(x) \vee f(y) = f(x \vee y) \in J$

Therefore $x \vee y \in f^{-1}(J)$

Now if $x \in f^{-1}(J)$ and $r \in A_1$

Then $f(x) \in J$, $f(r) \in A_2$ and as J is an ideal of A_2 , therefore

$f(x) \wedge f(r) = f(x \wedge r) \in J$

Thus $x \wedge r \in f^{-1} (J)$ therefore $f^{-1}(J) = \{ x \in A1 / f(x) \in J \subseteq A_2 \}$ is an ideal of A_1

2) Let J is filter of A_2 and $f^{-1}(J)= \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$

Let $x , y \in f^{-1}(J)$ then $f(x) , f(y) \in J$

As J is filter , $f(x) \wedge f(y) = f(x \wedge y) \in J$

Therefore $x \wedge y \in f^{-1}(J)$

Now if $x \in f^{-1} (J)$ and $r \in A_1$

Then $f(x) \in J$, $f(r) \in A_2$ and as J is filter of A_2 ,therefore

$f(x) \vee f(r) = f(x \vee r) \in J$

Thus $x \vee r \in f^{-1}(J)$ therefore $f^{-1}(J) = \{ x \in A1 / f(x) \in J \subseteq A_2 \}$ is filter of A_1

Theorem: III.6:

Let A_1 and A_2 be two distributive q-lattices and f: $A_1 \rightarrow A_2$ be a onto homomorphism then

1) $f(P^*) \subseteq (f(P))^*$

Proof: let $x \in f(P:I)$

Implies there exits $y \in (P:I)$ such that $f(y) = x \in f(P:I)$

Implies $y \wedge a \in I$ for all $a \in P$

Implies $f(y \wedge a) \in f(I)$ for all $a \in P$

Implies $f(y) \wedge f(a) \in f(I)$ for all $a \in P$ that is for all $f(a) \in f(P)$

Implies $f(y) \in \{ f(P) : f(I)\}$

Implies $x \in \{ f(P) : f(I)\}$

Implies $f(P:I) \subseteq \{ f(P) : f(I)\}$

Therefore $f(P^*) \subseteq (f(P))^*$

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