Exact vacuum solutions of six dimensional Bianchi Type-I space-time in $f(R)$ theory of gravity

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Abstract

In this paper, we have extended the five dimensional work refer it to [2] regarding the accelerating expansion of the universe to higher six dimension and obtained exact vacuum solutions of six dimensional Bianchi type-I space time in $f(R)$ theory of gravity using metric approach. In particular, two different models of the universe have been investigated using the law of variation of Hubble parameter. We observed that, the first model is singular and second one is non singular. The physical properties of these models have been discussed and evaluated function of Ricci scalar, $f(R)$ for both the models. It is interesting to note that, our five dimensional work explained in [2] and the work of M. Sharif and M. Farasat Shamir ((2009) in V_4 regarding the universe expansion can be reproduced by reducing the dimensions.

KEYWORDS: $f(R)$ theory of gravity, six dimensional Bianchi Type-I space-time, vacuum field equations in V_6.

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§ 1. Introduction

In the paper [2], we have discussed the well-known phenomenon of the universe expansion in five dimensional Bianchi type-I space time using the vacuum field equations in $f(R)$ theory of gravity and obtained exact vacuum solutions with the help of metric approach on the lines of M. Sharif and M. Farasat Shamir (2009). In particular, we have obtained two exact solutions using the variation law of Hubble parameter. These solutions correspond to two models of the universe in V_5. It has been observed that, the first solution $(n \neq 0)$ gives a singular model while the second $(n = 0)$ provides a non-singular model. The physical behaviour of these five dimensional Bianchi type-I space time models has also been discussed. Moreover, the function $f(R)$ of the Ricci scalar has been evaluated for both the models. We observed that the five dimensional work regarding the accelerating expansion of the universe can further be extended to higher six dimensional Bianchi type-I space-time and therefore an attempt has been made in the present paper.

Thus in the present paper, we propose to find exact vacuum solutions of six dimensional Bianchi type-I space-time in $f(R)$ theory of gravity using the metric approach.

The corresponding field equations of $f(R)$ gravity theory in V_6 are given by

$$F(R)R_{ij} - \frac{1}{2} f(R)g_{ij} - \nabla_i \nabla_j F(R) + g_{ij} \Box F(R) = kT_{ij}, \quad (i, j = 1,2,3,4,5,6) \tag{1}$$

where $F(R) \equiv \frac{df(R)}{dR}$, $\Box \equiv \nabla^i \nabla_i$ (2)

with ∇_i the covariant derivative and T_{ij} is the standard matter energy momentum tensor. These are the fourth order partial differential equations in the metric tensor. The fourth order is due to the last two terms on the left hand side of the equation. If we consider $f(R) = R$, these equations of $f(R)$ theory of gravity reduce to the field equations of Einstein's general theory of relativity in V_6.

After contraction of the field equation (1), we get

$$F(R)R - 3f(R) + 5\Box F(R) = kT. \tag{3}$$

In vacuum this field equation (3) reduces to

$$F(R)R - 3f(R) + 5\Box F(R) = 0. \tag{4}$$

This gives a relationship between $f(R)$ and $F(R)$ which can be used to simplify the field equations and to evaluate $f(R)$.

The paper is organized as follows: Sections 2 is used to find exact vacuum solutions of six dimensional Bianchi type-I space-time and the singularity analysis of these solutions. Section 3 and 4 are dealt with six dimensional models of the universe and in the last section, we summarize and conclude the results.

§ 2 Exact Vacuum Solutions of Six dimensional Bianchi Type - I Space-Time

In this section we propose to find exact vacuum solutions of six dimensional Bianchi Type-I space time in $f(R)$ gravity.

The line element of the six dimensional Bianchi type-I space time is given by

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)(dz^2 + du^2 + dv^2) \tag{5}$$

where A, B and C are cosmic scale factors. The corresponding Ricci scalar becomes

$$R = -2[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{3\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{3\dot{B}\dot{C}}{BC} + \frac{3\dot{A}\dot{C}}{AC} + \frac{3\dot{C}^2}{C^2}] \tag{6}$$

where dot denotes the derivative with respect to t.

We define the average scale factor a as

$$a = (ABC^3)^{\frac{1}{5}}, \tag{7}$$

and the volume scale factor is defined as

$$V = a^5 = ABC^3. \tag{8}$$

The generalized mean Hubble parameter H is defined by

$$H = \frac{1}{5}\sum_{i=1}^{5} H_i \tag{9}$$

where $H_1 = \frac{\dot{A}}{A}$, $H_2 = \frac{\dot{B}}{B}$, $H_3 = H_4 = H_5 = \frac{\dot{C}}{C}$ are the directional Hubble parameters in the directions of x, y, z, u and v axes respectively. Using equations (7), (8) and (9), we obtain

$$H = \frac{1}{5}\frac{\dot{V}}{V} = \frac{1}{5}[H_1 + H_2 + H_3 + H_4 + H_5] = \frac{\dot{a}}{a}. \tag{10}$$

From equation (4), we have

$$f(R) = \frac{1}{3}\{5\,\Box F(R) + F(R)R\}. \tag{11}$$

Putting this value of $f(R)$ in the vacuum field equations (3), we obtain

$$\frac{F(R)R_{ij} - \nabla_i \nabla_j F(R)}{g_{ij}} = \frac{1}{6}[F(R)R - \Box F(R)]. \tag{12}$$

Since the metric (5) depends only on t, one can view equation (12) as the set of differential equations for $F(t)$, A, B and C. It follows from equation (12) that the combination

$$A_i = \frac{F(R)R_{ii} - \nabla_i \nabla_i F(R)}{g_{ii}}, \tag{13}$$

is independent of the index i and hence $A_i - A_j = 0$ for all i and j. Consequently $A_6 - A_1 = 0$ gives

$$-\frac{\ddot{B}}{B} - \frac{3\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{3\dot{A}\dot{C}}{AC} + \frac{\dot{A}\dot{F}}{AF} - \frac{\ddot{F}}{F} = 0, \tag{14}$$

From $A_6 - A_2 = 0$, $A_6 - A_3 = 0$, $A_6 - A_4 = 0$ and $A_6 - A_5 = 0$, we have

$$-\frac{\ddot{A}}{A} - \frac{3\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{3\dot{B}\dot{C}}{BC} + \frac{\dot{B}\dot{F}}{BF} - \frac{\ddot{F}}{F} = 0, \tag{15}$$

$$-\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{2\ddot{C}}{C} + \frac{2\dot{C}^2}{C^2} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{F}}{CF} - \frac{\ddot{F}}{F} = 0, \tag{16}$$

It is noted that here, we obtain three independent non-linear differential equations with four unknowns namely A, B, C and F.

Now (14)-(15), (15)- (16) and (14)- (16), we get respectively

$$\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}+\frac{3\dot{C}}{C}(\frac{\dot{A}}{A}-\frac{\dot{B}}{B})+\frac{\dot{F}}{F}(\frac{\dot{A}}{A}-\frac{\dot{B}}{B})=0\,, \tag{17}$$

$$\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}-\frac{2\dot{C}^2}{C^2}+\frac{\dot{A}\dot{B}}{AB}+\frac{2\dot{B}\dot{C}}{BC}-\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{F}}{F}(\frac{\dot{B}}{B}-\frac{\dot{C}}{C})=0\,, \tag{18}$$

$$\frac{\ddot{A}}{A}-\frac{\ddot{C}}{C}-\frac{2\dot{C}^2}{C^2}+\frac{\dot{A}\dot{B}}{AB}+\frac{2\dot{A}\dot{C}}{AC}-\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{F}}{F}(\frac{\dot{A}}{A}-\frac{\dot{C}}{C})=0\,. \tag{19}$$

After integration the above equations imply that

$$\frac{B}{A}=d_1\exp[c_1\int\frac{dt}{a^5F}], \tag{20}$$

$$\frac{C}{B}=d_2\exp[c_2\int\frac{dt}{a^5F}], \tag{21}$$

$$\frac{A}{C}=d_3\exp[c_3\int\frac{dt}{a^5F}] \tag{22}$$

where c_1, c_2, c_3 and d_1, d_2, d_3 are constants of integration which satisfied the relation

$c_1+c_2+c_3=0$ and $d_1d_2d_3=1$.

From equations (20) -(22), the metric functions are obtained explicitly as

$$A=ap_1\exp[q_1\int\frac{dt}{a^5F}], \tag{23}$$

$$B=ap_2\exp[q_2\int\frac{dt}{a^5F}], \tag{24}$$

$$C=ap_3\exp[q_3\int\frac{dt}{a^5F}], \tag{25}$$

where $p_1=(d_1^{-4}d_2^{-3})^{1/5}$, $p_2=(d_1d_2^{-3})^{1/5}$, $p_3=(d_1d_2^{2})^{1/5}$, (26)

and

$$q_1=-\frac{4c_1+3c_2}{5}, \quad q_2=\frac{c_1-3c_2}{5}, \quad q_3=\frac{c_1+2c_2}{5}. \tag{27}$$

We have pointed out that p_1, p_2, p_3 and q_1, q_2, q_3 are related by

$$p_1 p_2 p_3^3 = 1, \qquad q_1 + q_2 + 3q_3 = 0\,. \tag{28}$$

Now we use the power law assumption to solve the integral part in the above equations as

$$F \,\alpha\, a^m\,, \tag{29}$$

where m is an arbitrary constant.

The equation (29) implies that

$$F = ka^m \tag{30}$$

where k is the constant of proportionality and m is any integer.

The deceleration parameter q in cosmology is the measure of the cosmic acceleration of the universe expansion and is defined as

$$q = -\frac{a\ddot{a}}{\dot{a}^2}\,. \tag{31}$$

The sign of q plays an important role to identify the behaviour of the universe. The positive deceleration parameter corresponds to a decelerating model while the negative value provides inflation.

The well-known relation between the average Hubble parameter H and average scale factor a given as

$$H = la^{-n} \tag{32}$$

where $l > 0$ and $n \geq 0$.

From equation (10) and (32), we have

$$\dot{a} = la^{1-n} \tag{33}$$

and consequently the deceleration parameter becomes

$$q = n - 1 \tag{34}$$

which is a constant. After integrating equation (34), we have

$$a = (nlt + k_1)^{1/n}, \quad n \neq 0 \tag{35}$$

and

$$a = k_2 \exp(lt), \quad n = 0, \tag{36}$$

where k_1 and k_2 are constants of integration.

Thus we have two values of the average scale factors which correspond to two different models of the universe.

From the scalar R given in the equation (6), we can check the singularity of the solutions. If we consider $m = -4$ as a special case then from equation (30) we have

$$F = ka^{-4} \tag{37}$$

and

$$R_1 = -\frac{2}{a^2k^2}[5k^2(a\ddot{a} + 2\dot{a}^2) - 9(q_1q_3 + q_2q_3 + \frac{q_1q_2}{9} + \frac{10}{3}q_3^2)] \tag{38}$$

which shows that singularity occurs at $a = 0$.

§ 3. Six dimensional Model of the Universe when $n \neq 0$

In this section we study the six dimensional model of the universe for $n \neq 0$.

When $n \neq 0$ then we have $a = (nlt + k_1)^{1/n}$ and for $m = -4$ as a special case, F becomes

$$F = k(nlt + k_1)^{-4/n}. \tag{39}$$

For this value of F equations (23), (24) and (25) imply that

$$A = p_1(nlt + k_1)^{1/n} \exp[\frac{q_1(nlt + k_1)^{\frac{n-1}{n}}}{kl(n-1)}], n \neq 1 \tag{40}$$

$$B = p_2(nlt + k_1)^{1/n} \exp[\frac{q_2(nlt + k_1)^{\frac{n-1}{n}}}{kl(n-1)}], n \neq 1 \tag{41}$$

$$C = p_3(nlt + k_1)^{1/n} \exp[\frac{q_3(nlt + k_1)^{\frac{n-1}{n}}}{kl(n-1)}], \quad n \neq 1. \tag{42}$$

The mean generalized Hubble parameter becomes

$$H = \frac{l}{nlt + k_1}. \tag{43}$$

And the volume scale factor becomes

$$V = (nlt + k_1)^{5/n}. \tag{44}$$

From the equation (11), the function $f(R)$ found as

$$f(R) = \frac{k}{3}(nlt + k_1)^{-\frac{4}{n}} R + \frac{20}{3} kl^2 (n-1)(nlt + k_1)^{-\frac{4}{n}-2}. \tag{45}$$

From equation (38) Ricci scalar $R = R_1$ becomes

$$R = R_1 = -2[l^2(15-5n)(nlt + k_1)^{-2} - \frac{(q_1q_2 + 3q_2q_3 + 3q_1q_3 + 3q_3^2)}{k^2}(nlt + k_1)^{-2/n}], \tag{46}$$

which clearly indicates that $f(R)$ cannot be explicitly written in terms of R. However, by inserting this value of R, $f(R)$ can be written as a function of t, which is true as R depends upon t. For a special case when $n = 1/2$, $f(R)$ turns out to be

$$f(R) = \frac{k}{3}[\frac{-25l^2 \pm \sqrt{625l^2 + 8(q_1q_2 + 3q_2q_3 + 3q_1q_3 + 3q_3^2)R/k^2}}{2R}]^{-4} R$$

$$-\frac{10kl^2}{3}[\frac{-25l^2 \pm \sqrt{625l^4 + 8(q_1q_2 + 3q_2q_3 + 3q_1q_3 + 3q_3^2)R/k^2}}{2R}]^{-5}. \tag{47}$$

This gives $f(R)$ only as a function of R.

§ 4. Six dimensional Model of the Universe when $n = 0$

In this section, we study the six dimensional model of the universe for $n = 0$.

For $n = 0$ the average scale factor for the model of the universe is $a = k_2 \exp(lt)$ and hence F becomes

$$F = \frac{k}{k_2^4}\exp(-4lt). \tag{48}$$

For this value of F equations (23), (24) and (25), imply that

$$A = p_1 k_2 \exp(lt) \exp[-\frac{q_1 \exp(-lt)}{klk_2}], \tag{49}$$

$$B = p_2 k_2 \exp(lt) \exp[-\frac{q_2 \exp(-lt)}{klk_2}], \tag{50}$$

$$C = p_3 k_2 \exp(lt) \exp[-\frac{q_3 \exp(-lt)}{klk_2}]. \tag{51}$$

The mean generalized Hubble parameter becomes

$$H = l. \tag{52}$$

And the volume scale factor becomes

$$V = k_2^5 \exp(5lt). \tag{53}$$

From the equation (11), the function $f(R)$ found as

$$f(R) = \frac{k}{3k_2^4} \exp(-4lt)(R - 20l^2). \tag{54}$$

From equation (38) Ricci scalar $R = R_1$ becomes

$$R = R_1 = -2[20l^2 - \frac{q_1 q_2 + 3q_2 q_3 + 3q_1 q_3 + 3q_3^2}{k^2 k_2^2 \exp(2lt)}]. \tag{55}$$

Here we can get the general function $f(R)$ in terms of R

$$f(R) = \frac{k^5}{3(q_1 q_2 + 3q_2 q_3 + 3q_1 q_3 + 3q_3^2)^2} (R - 20l^2)(R + 30l^2)^2. \tag{56}$$

which corresponds to the general function $f(R)$

$$f(R) = \sum a_n R^n \tag{57}$$

where n may take the values from negative or positive.

§ 5 Concluding Remark

In the paper [1], M. Sharif and M. Farasat Shamir (2009) have investigated two exact vacuum solutions of the four dimensional Bianchi type-I space time in $f(R)$ theory of gravity by using the variation law of Hubble parameter to discuss the well-known phenomenon of the universe expansion. This work regarding exact vacuum solutions in $f(R)$ theory of gravity has further been extended to five dimensional Bianchi type-I space time in our paper refer it to [1].

In this paper, we have extended the study regarding exact vacuum solutions in $f(R)$ theory of gravity to six dimensional Bianchi type-I space time and obtained two exact vacuum solutions corresponding to two models of the universe (i.e. $n \neq 0$ and $n = 0$). The first solution gives a singular model with power law expansion and positive deceleration parameter while the second solution gives a non-singular model with exponential expansion and negative deceleration parameter. The functions $f(R)$ are evaluated for both models.

The physical behavior of these six dimensional models is observed as under :

i. For $n \neq 0$ i.e. Singular six dimensional model of the universe

For this model average scale factor $a = (nlt + k_1)^{1/n}$.

This model has point singularity at $t = -k_1 / nl$.

The physical parameters H_1, H_2, H_3, H_4, H_5 and H are all infinite at this point of singularity.

The volume scale factor V vanishes at this point.

The function of the Ricci scalar, $f(R)$ is also infinite at this point.

The metric functions A, B and C vanish at this point of singularity.

In this way we can conclude from these observations that the six dimensional model of the universe starts its expansion with zero volume at $t = -\frac{k_1}{nl}$ *and it continues to expand.*

ii. For $n = 0$ i.e. non-singular six dimensional model of the universe

For this model average scale factor $a = k_2 \exp(lt)$.

This six dimensional model of the universe is non-singular because exponential function is never zero and hence there does not exist any physical singularity for this six dimensional model of the universe.

The physical parameters H_1, H_2, H_3, H_4 and H_5 are all finite for all finite values of t.

The mean generalized Hubble parameter H is constant.

The function of the Ricci scalar, $f(R)$ is also finite.

The metric functions A, B and C do not vanish for this model.

The volume scale factor increases exponentially with time which indicates that the six dimensional model of the universe starts its expansion with zero volume from infinite past.

The solution obtained here is more general to that of earlier exact vacuum solutions obtained in V_4 and V_5.

It is shown that results regarding accelerating expansion of the universe obtained in six dimensional Bianchi type -I space time are similar to those of V_4 and V_5, essentially retaining their mathematical format.

It is pointed out that the work of M. Sharif and M. Farasat Shamir (2009) in V_4 and our work in V_5 regarding accelerating expansion of the universe emerge as special cases of our work carried out in the present paper.

It has been observed that, exact solutions in $f(R)$ theory of gravity concerning the non-vacuum Einstein's modified field equations in six dimensional Bianchi type-I space-time can further be investigated.

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