Even Vertex Gracefulness of Book B_n when n is Odd

Manisha M. Acharya
Associate Professor of Mathematics & Head of department of Mathematics, M.D. College, Mumbai, Maharashtra, India

Abstract

Labeling of a graph G is an assignment of integers either to the vertices of G or edges of G or both subject to certain conditions. The labeling is considered as an Injective map either from set of vertices of G to a set of integers or from set of edges of G to a set of integers. A graph is Even vertex graceful if there exists an injective map f : E (G) → { 1,2,---- 2q} so that the induced map f^+ :V(G)→ {0,2,4 --- 2k–2} defined by $f^+ (x) = \sum f(xy) \pmod{2k}$ where the sum runs over all edges xy through y and k = max(p,q) be such that all vertices get distinct labels. In this paper it is proven that Book B_n is even vertex graceful when n is odd.

KEYWORDS – Book, Even vertex gracefulness, Induced Vertex labeling, Labeling of graphs.

Introduction

Let G = (V, E) be a simple graph with a finite non empty set V of 'p' vertices together with set E of q unordered pairs of distinct points of V. Each pair $e = (u_1,u_2)$ of points in E is an edge of G.A graph with p vertices and q edges is called a (p,q) graph. A graph is said to be of order p. [4]

Definition : A map f:V(G) → {0,1,2,----q} is called a graceful labeling if f is one – to – one and the edges receive all the labels from 1 to q where the label of an edge is the absolute value of the difference between vertex labels at its ends. A graph having a graceful labeling is called a graceful graph. [2]

Definition : A graph is Even vertex graceful if there exists an injective map f : E (G) → { 1,2,---- 2q} so that the induced map f^+ V(G)→ {0,2,4 --- 2k–2} defined by $f^+ (x) = \sum f(xy) \pmod{2k}$ where the sum runs over all edges xy through y and k = max(p,q) gives distinct labels to all vertices in G [5]

Definition: Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2. Then Cartesian products of G_1 and G_2 is denoted by G_1 x G_2. To define the product G_1 x G_2, consider any two points $u= (u_1 , u_2)$ and $v=(v_1 , v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in G_1x G_2 Whenever $u_1= v_1$ and u_2 adj v_2 or $u_2= v_2$ and u_1 adj v_1[4]

Definition : For $n \geq 3$ the Book B_n is the certesian product S_n X K_2 where S_n is the star with n end – vertices and K_2 is the complete graph with 2–vertices. [4]

The author has used the terminology and notations of Harary [4] .So , for terms not defined here and notations not explained here refer to Harary [4]

The author has also proven "Even vertex gracefulness of book $B_{n;}$ when n is even"[7]

MAIN RESULT :

Theorem : For an odd integer $n \geq 3$, book B_n is even vertex graceful.

Proof : Here B_n is a book with 'n' number of pages. It is an open book with (n+1)/2 number of pages on left handside and (n–1)/2 number of pages on right hand side.

Number of vertices in Bn = $|V(Bn)|$ = p =2 n + 2. Number of edges in Bn = $|E(Bn)|$ =q= 3n+1

Let V (Bn) = $\{a_1, a_2, - --- , a_n, b_1, b_2 ---- b_n, A, B\}$ and E(Bn) = $\{e_1, e_2 ---- , e_{3n+1}\}$
The middle edge of the first page is numbered as e_1, then lower edge by e_2 and upper edge by e_3. The first(n – 1)/2 pages contain edges numbered with $\{e_1, e_2, e_3 ---- e_{3n-3}\}$

and last (n – 3)/2 pages contain edges numbered with $\{e_{(3n-1)/2}, e_{(3n+1)/2}, \ldots, e_{3n-6}\}$

The middle two pages contain edges $\{e_{3n-5}, e_{3n-4}, --- e_{3n}\}$. The middle edge of the book is numbered as e_{3n+1}

Upper end vertices of pages are numbered as a_1, a_2 --- a_n and vertices at the lower end of pages are numbered as b_1, b_2---, b_n such that edges incident at a_i are (e_{3i-2}, e_{3i}) and edges incident at b_i are (e_{3i-2}, e_{3i-1}) for all i; $1 \leq$, $i \leq n$. The vertex where all edges e_{3i} , $(1 \leq i \leq n)$ and e_{3n+1} meet is denoted by A. The vertex where all e_{3i-1}, $(1 \leq i \leq n)$ and e_{3n+1} meet is denoted by B.

Figure 1 shows numbering of Book B_7

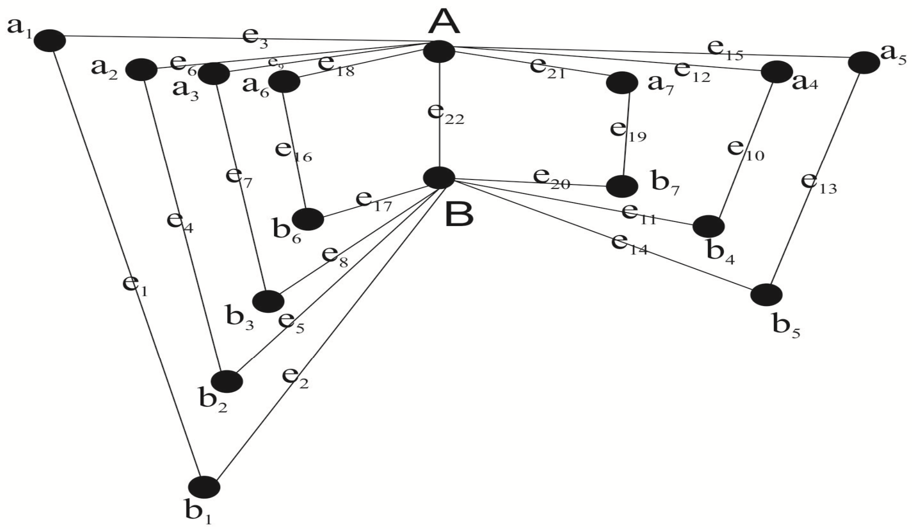

Figure 1

Let the map f :E(Bn) → { 1,2 , --- 6n + 2} denote labeling of edges Define $f(e_{3n+1})$ = 1 and $f(e_i) = 2i + 1$ for $1 \leq i \leq 3n-6$ Then the induced map $f^+(u) = \sum f(uv)$ (mod 6n+2) where the sum runs over all edges uv through v defines vertex labeling. Clearly distinct labels are used for edges e_1, e_2---- e_{3n-6} and e_{3n+1}. They are 3,5,7, --- , 6n-11 and 1 respectively. Hence induced vertex labels for a_1, a_2, ---, a_{n-2} are 10, 22,..,12n – 50, 12n – 38, 12n – 26 (mod 6n+2) respectively.

Similarly induced vertex labels for b_1,b_2,… b_{n-3}, b_{n-2} are 8,20, ……, 12n–52, 12n – 40,12 n – 28 (mod 6n+2) respectively.

For the labeling of remaining six edges namely e_{3n-5} e_{3n-4}, e_{3n-3}, e_{3n-2}, e_{3n-1}, and e_{3n}, three different cases are considered viz: $n \equiv 1,3,5 \pmod 6$

The case : n ≡1 (mod 6)

For first three edges, that is, e_{3n-5}, e_{3n-4} and e_{3n-3} The function f is $f(e_i) = 2i + 1$ for $3n-5 \leq i \leq 3n-3$

and for remaining three edges, a function is defined as $f(e_{3n-2}) = 6n + 1$;

$f(e_{3n-1}) = 6n-3; f(e_{3n}) = 6n - 1$

Hence induced vertex labels for a_{n-1}, a_n are 12n–14,12n (mod 6n +2) respectively.Also, induced vertex labels for b_{n-1}, b_n are 12n-16, 12n-2, (mod 6n+2) respectively.

Lastly to determine induced vertex labeling for A and B. As mentioned earlier, edges incident at A are e_{3i} for $1 \leq i \leq n$ and edge e_{3n+1}

Hence Induced vertex Label for A

$$\equiv [f(e_3) + f(e_6) + \ldots + f(e_{3n-3})] + f(e_{3n}) + f(e_{3n-1}) \pmod{6n+2}$$

$$\equiv [7 + 13 + 19 + \ldots + (6n-5)] + (6n-1) + 1 \pmod{6n+2}$$

$$\equiv 3n^2 + 4n - 1 \pmod{6n+2}$$

$$\equiv 6n \pmod{6n+2}$$

Hence Induced vertex label for A = 6n

Similarly we calculate Induced vertex labeling for B.

The edges incident at B are e_{3i-1} for $1 \leq i \leq n$ and e_{3n+1}

Then induced vertex label for B

$$\equiv [f(e_2) + f(e_5) + f(e_8) + \ldots + f(e_{3n-4})] + f(e_{3n-1}) + f(e_{3n-1}) \pmod{6n+2}$$

$$\equiv [5 + 11 + 17 + \ldots + (6n-7)] + (6n-3) + 1 \pmod{6n+2}$$

$$\equiv 3n^2 + 2n - 1 \pmod{6n+2}$$

$$\equiv 4n \pmod{6n+2}$$

Induced vertex label for B= 4n

The case : n ≡ 3 (mod 6)

The labeling for edges e_{3n-5}, e_{3n-4} and e_{3n-3} is similar to that of the case $n \equiv 1 \pmod 6$

For remaining three edges, a function is slightly different and defined as

$f(e_{3n-2}) = 6n–3$;$f(e_{3n-1}) = 6n + 1$;$f(e_{3n}) = 6n–1$

Hence induced vertex labels for a_{n-1}, a_n are 12n–14, 12n–4 (mod 6n+2) respectively.

Also induced vertex labels for b_{n-1}, b_n are 12n–16, 12n–2 (mod 6n+2) respectively.

One can observe that induced vertex labeling for b_{n-1} and a_{n-1} are same as that of earlier case,as the labeling of edges incident at these vertices are same as that of earlier case. Also induced vertex labeling for b_n remains same because edge labels for e_{3n-2} and e_{3n-1} are interchanged from that of earlier labeling.

Labeling of edges incident at vertex A is same as that of earlier case. Hence in this case also induced vertex label for A = 6 n

Lastly to find induced vertex labeling for B

Induced vertex label for $B \equiv [f(e_3)]+f(e_5)+\ldots.+f(e_{3n-4})]+f(e_{3n-1})+f(e_{3n})(\text{mod} 6n+2)$

$$\equiv 3n^2 + 2n +3 \pmod{6n + 2}$$

$$\equiv 4n +4 \pmod{6n +2}$$

Induced vertex label for B = 4n + 4

<u>The case</u> : n ≡5 (mod 6)

In this case a function f is defined as follows :

$f(e_{3n-5}) = 6n – 7$, $f(e_{3n-4}) = 6n – 9$, $f(e_{3n-3}) = 6n -5$, $f(e_{3n-2})= 6n+1$, $f(e_{3n-1}) = 6n-3$, $f(e_{3n}) = 6n – 1$

Induced vertex labels for a_{n-1} and a_n are 12n–12 and 12n + 1 (mod 6n + 2) respectively.

Also induced vertex labels for b_{n-1}, b_n are 12n–16,12n–2 (mod 6n + 2) respectively.

Lastly to calculate induced vertex labels for A and B

Induced vertex label of A

$\equiv [f(e_3) + f(e_6) + \ldots. + f(e_{3n-6})] + [f(e_{3n-3}) + f[(e_{3n})] + [f(e_{3n+1})] \pmod{6n + 2}$

=6 n

Similarly induced vertex label of B

$\equiv [f (e_2) + f (e_5) + \ldots.. + f(e_{3n-7})] + [f(e_{3n-4}) + f (e_{3n-1})]+ f (e_{3n+1}) \pmod{6n + 2}$

= 4n-2

<u>Illustration</u> :

Figure 2 shows Even Vertex Gracefulness of Book B_9 (The case n ≡ 3 (mod 6))

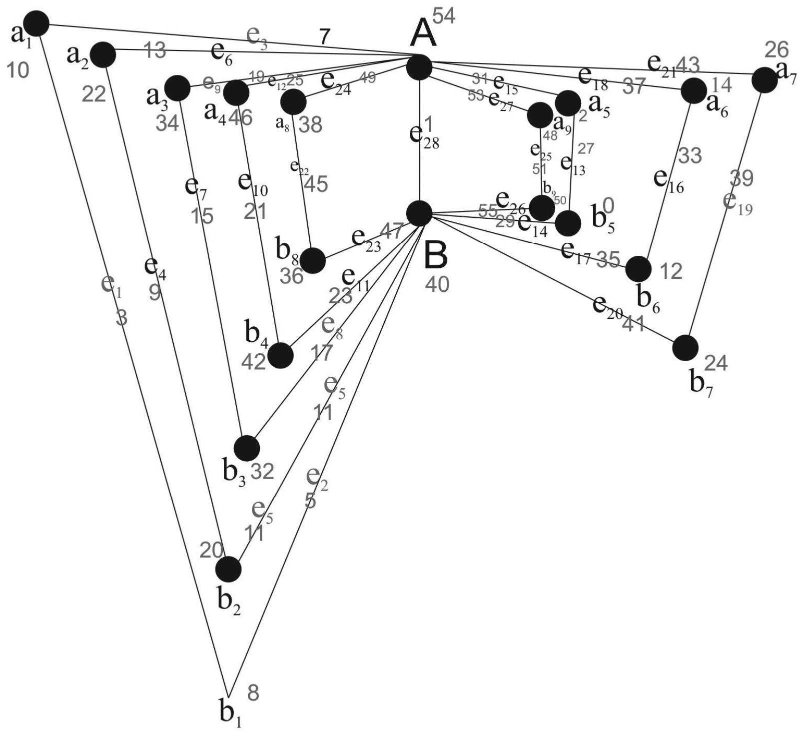

Figure 2

Conclusion: Labels assigned to edges e_1,e_2---- e_{3n-6} and e_{3n+1} are 3,5,7, --- , 6n-11 and 1 respectively.

Hence induced vertex labels for a_1, a_2, ---, a_{n-2} are 10, 22,….,12n – 50, 12n – 38,

12n – 26 (mod 6n+2) respectively.

Similarly induced vertex labels for $b_1,b_2,\ldots$ b_{n-3}, b_{n-2} are 8,20, ……, 12n–52,

12n – 40,12n – 28 (mod 6n+2) respectively.

For the remaining six edges e_{3n-5} to e_{3n} we considered three cases. In all three cases edge labels assigned are 6n-9 ,6n-7,6n-5,6n-3,6n-1,6n+1 .

Hence induced vertex labels for a_{n-1},a_n,b_{n-1},b_n,A and B in three different cases are respectively as follows:

In the case n≡1(mod 6) vertex labels are 12n-14,12n,12n-6,12n-2,6n and 4n (mod 6n+2)

In the case n≡3(mod6) **v**ertex labels are 12n-14,12n-4,12n-16,12n-2,6n and4n +4(mod 6n+2)

In the case n ≡5 (mod 6) vertex labels are 12n-12,12n+1,12n-16,12n-2,6n and

4n -2(mod 6n+2)

Therefore f and f^+ satisfy even vertex graceful labeling. Hence when n is odd book B_n is even vertex graceful .

References:

1) C. Barrientos and H. Hevia,On 2- Equitable Labelings Of Graphs, Research supported in part by FONDECYT project 1941219(94)

2)C. Barrientos, Odd-Graceful Labelings Of Trees Of Diameter 5,AKCE J. Graphs. Combin,6,No.2(2009)

3)Gallian, J.A. 2009. A Dynamic Survey of Graph Labeling. The Electronic Journal of Combinatorics. (Jan 31, 2009).

4)Harary,Graph Theory, Addison-Wesley,1968.

5) A. Solairaju and P. Murugananthan. Even vertex graceful of Path,Circuit, Star, Wheel,some Extension-friendship Graphs and Helm Graph, International Journal of Computer Applications(0975-8887),vol.10-No.6,Nov.2010

6) A.Solairaju and P. Muruganantham, Even vertex gracefulness of fan graph, International Journal of Computer Applications (0975-8887): Vol, 8. No. 8, (October, 2012).

7)Manisha Acharya ,Even vertex gracefulness of book B_n when n is even , sent for publication