Characteristic Exponent for the Stability of L_5 in Photo-Gravitational Restricted Problem of 2+2 Bodies

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Abstract

Stability of L_5 has been examined by finding characteristic roots in 2+2 body problem when one of the primaries is a radiating body. It has been found that stability of L_5 depends on the value of Ɛ and L_5 is stable for $\varepsilon < 6.67551 \times 10^{-11}$ when $0 < \mu \leq 0.3$ and $\mu_1 = \mu_2 = 10^{-12}$.
KEYWORDS: Stability, 2+2 body problem, Photo-Gravitation

1. INTRODUCTION

Equilibrium solutions of restricted problem of 2+2 bodies are derived by Whipple (1984) in which M_1 and M_2 are two point masses moving in a circular Keplerian orbit about their centre of mass. He assumed that $M_1 \geq M_2$. Two minor bodies (m_1, $m_2 << M_2$) move in the gravitational field of primaries (M_1 and M_2). They attract each other but do not perturb the primaries.

Radizievskii (1950) used the idea of Photo-Gravitation for three specific bodies- the sun, a planet and a dust particle. He showed that an allowance of direct solar radiation pressure results a change in the position of libration point.

Here, we consider that bigger primary is a radiating body. Hence, by applying Photo-Gravitation effect in Whipple's model, we get a new configuration.

2. EQUATION OF MOTION AND EQUILIBRIUM POINT L_5

Whipple's Equations of motion are:

$$\ddot{x}_i - 2\dot{y}_i = \frac{\partial T}{\partial x_i}, \quad \ddot{y}_i + 2\dot{y}_i = \frac{\partial T}{\partial y_i} \quad and \quad \ddot{z}_i = \frac{\partial T}{\partial z_i}, \quad (i = 1, 2) \tag{1}$$

Where, $T = \sum_{i=1}^{2} \mu_i \left[\frac{1}{2}(x_i^2 + y_i^2) + \frac{1-\mu}{r_{1i}} + \frac{\mu}{r_{2i}} + \frac{\mu_{3-i}}{2r}\right]$,

$\mu = \frac{M_2}{M_1 + M_2}$, $\mu_i = \frac{m_i}{M_1 + M_2}$, $(i = 1, 2)$, $r_{1i}^2 = (x_i - \mu)^2 + y_i^2 + z_i^2$,

$r_{2i}^2 = (x_i + 1 - \mu)^2 + y_i^2 + z_i^2$ and $r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$.

Here, we consider M_1 to be radiating one and we replace T by U as follow:

$$U = \sum_{i=1}^{2} \mu_i \left[\frac{1}{2}(x_i^2 + y_i^2) + \frac{q(1-\mu)}{r_{1i}} + \frac{\mu}{r_{2i}} + \frac{\mu_{3-i}}{2r}\right], \quad q = 1 - \varepsilon. \tag{2}$$

Now, equation of motion changes to

$$\ddot{x}_i - 2\dot{y}_i = \frac{\partial U}{\partial x_i}, \quad \ddot{y}_i + 2\dot{y}_i = \frac{\partial U}{\partial y_i} \quad and \quad \ddot{z}_i = \frac{\partial U}{\partial z_i}, \quad (i = 1, 2). \tag{3}$$

Equilibrium points of the system are those points where –

$$\dot{x}_i = \dot{y}_i = \dot{z}_i = \frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial y_i} = \frac{\partial U}{\partial z_i} = 0, \quad (i = 1, 2). \tag{4}$$

Garain (2013) found four values of L_5 in which two values $(x_i, y_i, 0)$, $(i = 1, 2)$ are known as in- line triangular equilibrium points and other two values $(x_i, y_i, 0)$, $(i = 1, 2)$ are known as perpendicular triangular equilibrium points of L_5.

3. CHARACTERISTIC EXPONENT FOR THE STABILITY OF L_5

Similar to Whipple (1984), the Characteristic equation corresponding to the point L_5 $(x_1, y_1, 0)$ is

$$f(\lambda) = \lambda^4 + \lambda^2\left[4 - \frac{1}{\mu_1}(U_{x_1x_1} + U_{y_1y_1})\right] + \frac{1}{\mu_1^2}\left[U_{x_1x_1}U_{y_1y_1} - (U_{x_1y_1})^2\right] = 0. \quad (5)$$

Let, $x_1 = P_{11} + \varepsilon Q_{11},\ x_2 = P_{21} + \varepsilon Q_{21},\ y_1 = R_{11} + \varepsilon S_{11},\ y_2 = R_{21} + \varepsilon S_{21}$ (6)

Where, $P_{11}, Q_{11}, R_{11}, S_{11}, P_{21}, Q_{21}, R_{21}$ *and* S_{21} are known values obtained from Garain (2013).

Let, $r_{11}^{-3} = a_1 + \varepsilon b_1$ (7)

Where,

$$a_1 = (P_{11}^2 + R_{11}^2 - 2P_{11}\mu + \mu^2)^{-\frac{3}{2}},\ b_1 = -3(P_{11}Q_{11} - Q_{11}\mu + R_{11}S_{11})(P_{11}^2 + R_{11}^2 - 2P_{11}\mu + \mu^2)^{-\frac{5}{2}}. \quad (8)$$

$r_{21}^{-3} = a_2 + \varepsilon b_2$ (9)

Where,

$$a_2 = \{P_{11}^2 + R_{11}^2 - 2P_{11}(1-\mu) + (1-\mu)^2\}^{-\frac{3}{2}},\ b_2 = -3\{P_{11}Q_{11} - Q_{11}(1-\mu) + R_{11}S_{11})\{P_{11}^2 + R_{11}^2 - 2P_{11}(1-\mu) + (1-\mu)^2\}^{-\frac{5}{2}}. \quad (10)$$

$r^{-3} = a_3 + \varepsilon b_3$ (11)

Where,

$$a_3 = \{(P_{11} - P_{21})^2 + (R_{11} - R_{21})^2\}^{-\frac{3}{2}},\ b_3 = -3\{(P_{11} - P_{21})(Q_{11} - Q_{21}) + (R_{11} - R_{21})(S_{11} - S_{21})\}\{(P_{11} - P_{21})^2 + (R_{11} - R_{21})^2\}^{-\frac{5}{2}}. \quad (12)$$

$r_{11}^{-5} = a_4 + \varepsilon b_4$ (13)

Where,

$$a_4 = (P_{11}^2 + R_{11}^2 - 2P_{11}\mu + \mu^2)^{-\frac{5}{2}},\ b_1 = -5(P_{11}Q_{11} - Q_{11}\mu + R_{11}S_{11})(P_{11}^2 + R_{11}^2 - 2P_{11}\mu + \mu^2)^{-\frac{7}{2}}. \quad (14)$$

$r_{21}^{-5} = a_5 + \varepsilon b_5$ (15)

Where,

$$a_5=\{P_{11}^2+R_{11}^2-2P_{11}(1-\mu)+(1-\mu)^2\}^{-\frac{5}{2}}, b_5=-5\{P_{11}Q_{11}-Q_{11}(1-\mu)+R_{11}S_{11}\}$$
$$\{P_{11}^2+R_{11}^2-2P_{11}(1-\mu)+(1-\mu)^2\}^{-\frac{7}{2}}. \quad (16)$$

$$r^{-5}=a_6+\varepsilon b_6 \quad (17)$$

Where,

$$a_6=\{(P_{11}-P_{21})^2+(R_{11}-R_{21})^2\}^{-\frac{5}{2}}, b_6=-5\{(P_{11}-P_{21})(Q_{11}-Q_{21})$$
$$+(R_{11}-R_{21})(S_{11}-S_{21})\}\{(P_{11}-P_{21})^2+(R_{11}-R_{21})^2\}^{-\frac{7}{2}}. \quad (18)$$

$$U_{x_1x_1}=u_{51}+\varepsilon V_{51} \quad (19)$$

Where,

$$u_{51}=\mu_1[1-a_1(1-\mu)-\mu a_2-\mu_2 a_3+3a_4(1-\mu)(P_{11}^2+\mu^2-2P_{11}\mu)$$
$$+3\mu a_5(P_{11}-\mu+1)^2+3\mu a_6(P_{11}-P_{21})^2], \quad (20)$$

$$V_{51}=\mu_1[(1-\mu)(b_1-a_1)-\mu b_2-\mu_2 b_3+6a_4 Q_{11}(1-\mu)(\mu-P_{11})+3(P_{11}^2+\mu^2-2P_{11}\mu)$$
$$(1-\mu)(b_4-a_4)+6a_6\mu_2(P_{11}-P_{21})(Q_{11}-Q_{21})+3\mu_2 b_6(P_{11}-P_{21})^2]. \quad (21)$$

$$U_{x_1y_1}=u_{52}+\varepsilon V_{52} \quad (22)$$

Where,

$$u_{52}=\mu_1[3(1-\mu)(a_4P_{11}R_{11}-\mu a_4R_{11})+3\mu(P_{11}-\mu+1)a_5R_{11}$$
$$+3\mu_2 a_6(P_{11}-P_{21})(R_{11}-R_{21})], \quad (23)$$

$$V_{52}=\mu_1[3(1-\mu)\{a_4Q_{11}R_{11}+a_4P_{11}S_{11}-a_4\mu S_{11}+(b_4-a_4)(P_{11}R_{11}-\mu R_1)\}$$
$$+3\mu(P_1-\mu+1)(a_5S_{11}+b_5R_{11}+a_5Q_{11}R_{11})+3\mu_2\{a_6(P_{11}-P_{21})(S_{11}-S_{21})$$
$$+a_6(Q_{11}-Q_{21})(R_{11}-R_{21})+b_6(P_{11}-P_{21})(R_{11}-R_{21})\}]. \quad (24)$$

$$U_{y_1y_1}=u_{53}+\varepsilon V_{53} \quad (25)$$

Where,

$$u_{53}=\mu_1[1-(1-\mu)a_1-\mu a_2-\mu_2 a_3+3(1-\mu)a_4R_{11}^2+3\mu a_5R_{11}^2+3\mu_2 a_6(R_{11}-R_{21})^2], \quad (26)$$

$$V_{53} = \mu_1[(1-\mu)(b_1 - a_1) - \mu b_2 - \mu_2 b_3 - 6(1-\mu)a_4 R_{11} S_{11} + 3(1-\mu)R_{11}^2(b_4 - a_4)$$

$$+6\mu a_5 R_{11} S_{11} + 3\mu b_5 R_{11}^2 + 6\mu_2 a_6(R_{11} - R_{21})(S_{11} - S_{21}) + 3\mu_2 b_6(R_{11} - R_{21})^2]. \quad (27)$$

The Characteristic equation f(λ) = 0 reduces to

$$f(\lambda) = \lambda^4 + A_{51}\lambda^2 + B_{51} = 0 \quad (28)$$

Where, $A_{51} = 4 - \frac{1}{\mu_1}(U_{x_1x_1} + U_{y_1y_1}) = a_{51} + \varepsilon b_{51}$ (29)

Here, $a_{51} = 4 - \frac{1}{\mu_1}(u_{51} + u_{53}),\ b_{51} = \frac{1}{\mu_1}(V_{51} + V_{53}).$ (30)

$B_{51} = \frac{1}{\mu_1^2}\{U_{x_1x_1} U_{y_1y_1} - (U_{x_1y_1})^2\} = a_{52} + \varepsilon b_{52}$ (31)

Where, $a_{52} = \frac{1}{\mu_1^2}(u_{51} u_{53} - u_{52}^2),\ b_{52} = \frac{1}{\mu_1^2}(u_{51} V_{53} + u_{53} V_{51} - 2u_{52} V_{52})$ (32)

Since, the equation (28) is quadratic in λ^2 and let, D be its discriminant, then

$$D = A_{51}^2 - 4B_{51} = C_{51} + D_{51} \quad (33)$$

Where, $C_{51} = a_{51}^2 - 4a_{52}$ and $D_{51} = 2a_{51} b_{51} - 4b_{52}$. (34)

Let, two roots of λ^2 in equation (28) be e_{51} and e_{52} respectively.

Where, $e_{51} = f_{51} + \varepsilon g_{51}$, (35)

$$f_{51} = \frac{1}{2}[\frac{1}{\mu_1}(u_{51} + u_{53}) - 4 + \sqrt{C_{51}}],\ g_{51} = \frac{1}{2}[\frac{1}{\mu_1}(V_{51} + V_{53}) + \frac{D_{51}}{2\sqrt{C_{51}}}] \quad (36)$$

$$e_{52} = f_{52} + \varepsilon g_{52}, \quad (37)$$

$$f_{52} = \frac{1}{2}[\frac{1}{\mu_1}(u_{51} + u_{53}) - 4 - \sqrt{C_{51}}],\ g_{52} = \frac{1}{2}[\frac{1}{\mu_1}(V_{51} + V_{53}) - \frac{D_{51}}{2\sqrt{C_{51}}}]. \quad (38)$$

We may get the value of λ^2 for given values of μ and μ_i (i =1, 2).

Table-1
Values of square of Characteristic roots
($\mu_1 = \mu_2 = 10^{-12}$, $e_{51} = f_{51} + \mathcal{E}\, g_{51}$, $e_{52} = f_{52} + \mathcal{E}\, g_{52}$)

μ	f_{51}	g_{51}	f_{52}	g_{52}
0.00001	-1312238.02	263368886501609000000000	-1316824.14	-263368886501609000000000
0.00002	-2625818.26	186228719522735000000000	-2632304.01	-186228719522735000000000
0.00003	-3939618.78	152055365956153000000000	-3947562.17	-152055365956153000000000
0.00004	-5253532.99	131684028807012000000000	-5262705.22	-131684028807012000000000
0.00005	-6567519.58	117781971752705000000000	-6577774.45	-117781971752705000000000
0.00006	-7881557.38	107519916516959000000000	-7892791.03	-107519916516959000000000
0.00007	-9195633.82	99544265001143400000000	-9207767.54	-99544265001143400000000
0.00008	-10509740.69	93115287850481900000000	-10522712.18	-93115287850481900000000
0.00009	-11823872.31	87790080648228800000000	-11837630.65	-87790080648228800000000
0.0001	-13138024.52	83285120194900100000000	-13152527.09	-83285120194900100000000
0.0002	-26280224.78	58892441337180800000000	-26300734.47	-58892441337180800000000
0.0003	-39423052.20	48086258901460600000000	-39448171.31	-48086258901460600000000
0.0004	-52566169.68	41644591562255100000000	-52595174.69	-41644591562255100000000
0.0005	-65709446.63	37248647389230800000000	-65741875.17	-37248647389230800000000
0.0006	-78852816.08	34003742261928900000000	-78888339.73	-34003742261928900000000
0.0007	-91996238.24	31481837661486400000000	-92034608.11	-31481837661486400000000
0.0008	-105139687.19	29449015260634400000000	-105180706.25	-29449015260634400000000
0.0009	-118283144.91	27765219914743100000000	-118326652.13	-27765219914743100000000
0.001	-131426598.27	26340793544692400000000	-131472458.85	-26340793544692400000000
0.002	-262859452.79	18628368982612600000000	-262924308.55	-18628368982614600000000
0.003	-394287332.52	15211864225002600000000	-394366763.20	-15211864225002600000000
0.004	-525709162.67	13175240663012400000000	-525800880.07	-13175240663012400000000
0.005	-657124521.57	11785316762030000000000	-657227063.33	-11785316762030000000000
0.006	-788533188.68	10759213954035000000000	-788645516.01	-10759213954035000000000
0.007	-919935029.46	9961603325570010000000	-920056355.19	-9961603325570010000000
0.008	-1051329953.2	9318532920501670000000	-1051459654.1	-9318532920501670000000
0.009	-1182717894.3	8785726255519190000000	-1182855460.9	-8785726255519190000000
0.01	-1314098802.4	8334836348399960000000	-1314243808.4	-8334836348399960000000
0.02	-2627514614.4	5887488260335790000000	-2627719655.6	-5887488260335790000000
0.03	-3940203269.8	4793185511751350000000	-3940454357.9	-4793185511751350000000
0.04	-5252152789.9	4131121787574720000000	-5252442680.9	-4131121787574720000000
0.05	-6563353330.9	3670134739368170000000	-6563677393.1	-3670134739368170000000
0.06	-7873795754.2	3321229757088840000000	-7874150695.7	-3321229757088840000000
0.07	-9183471262.1	3041964802340400000000	-9183854587.8	-3041964802340400000000
0.08	-10492371265.2	2809255120691220000000	-10492780998.4	-2809255120691220000000
0.09	-11800487323.4	2609369400658570000000	-11800921847.4	-2609369400658570000000
0.1	-13107811114.4	2433564006995210000000	-13108269075.7	-2433564006995210000000
0.2	-26135708142.1	1277513110525980000000	-26136354807.6	-1277513110525980000000
0.3	-39075845354.5	585361030952991000000	-39076636062.9	-585361030952991000000

It is clear from Table (1) that e_{52} is negative and the sign of e_{51} depends on the value of ε. For, $e_{51} < 0$, we must have $\varepsilon < \frac{|f_{51}|}{g_{51}}$.

4. CONCLUSION

In Table-1, we get that when $0 < \mu \leq 0.3$, $\mu_1 = \mu_2 = 10^{-12}$ and $\varepsilon < 6.67551 \times 10^{-11}$, the value of λ^2 is negative. So the value of each of the characteristic root will be purely imaginary and in this case L_5 is stable.

References

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