Characteristic Exponent for the Stability of L_4 in Photo-Gravitational Restricted Problem of 2+2 Bodies

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Abstract

Stability of L_4 has been examined by finding characteristic roots in 2+2 body problem when one of the primaries is a radiating body. It has been found that stability of L_4 depends on the value of ε. L_4 is stable for $\varepsilon < 5.16479\times 10^{-18}$.

KEYWORDS: Stability, 2+2 body problem, Photo-Gravitation

1. INTRODUCTION

Equilibrium solutions of restricted problem of 2+2 bodies are derived by Whipple (1984) in which M_1 and M_2 are two point masses moving in a circular Keplerian orbit about their centre of mass. He assumed that $M_1 \geq M_2$. Two minor bodies (m_1, $m_2 << M_2$) move in the gravitational field of primaries (M_1 and M_2). They attract each other but do not perturb the primaries.

Radizievskii (1950) used the idea of Photo-Gravitation for three specific bodies- the sun, a planet and a dust particle. He showed that an allowance of direct solar radiation pressure results a change in the position of libration point.

Here, we consider that bigger primary is a radiating body. Hence, by applying Photo-Gravitation effect in Whipple's model, we get a new configuration.

2. EQUATION OF MOTION AND EQUILIBRIUM POINT L_4

Whipple's Equations of motion are:

$$\ddot{x}_i - 2\dot{y}_i = \frac{\partial T}{\partial x_i}, \quad \ddot{y}_i + 2\dot{y}_i = \frac{\partial T}{\partial y_i} \quad and \quad \ddot{z}_i = \frac{\partial T}{\partial z_i}, \quad (i = 1, 2) \tag{1}$$

Where, $T = \sum_{i=1}^{2} \mu_i \left[\frac{1}{2}(x_i^2 + y_i^2) + \frac{1-\mu}{r_{1i}} + \frac{\mu}{r_{2i}} + \frac{\mu_{3-i}}{2r}\right]$,

$$\mu = \frac{M_2}{M_1 + M_2}, \quad \mu_i = \frac{m_i}{M_1 + M_2}, (i = 1, 2), \quad r_{1i}^2 = (x_i - \mu)^2 + y_i^2 + z_i^2,$$

$$r_{2i}^2 = (x_i + 1 - \mu)^2 + y_i^2 + z_i^2 \quad and \quad r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.$$

Here, we consider M_1 to be radiating one and we replace T by U as follow:

$$U = \sum_{i=1}^{2} \mu_i \left[\frac{1}{2}(x_i^2 + y_i^2) + \frac{q(1-\mu)}{r_{1i}} + \frac{\mu}{r_{2i}} + \frac{\mu_{3-i}}{2r}\right], \quad q = 1 - \varepsilon. \tag{2}$$

Now, equation of motion changes to

$$\ddot{x}_i - 2\dot{y}_i = \frac{\partial U}{\partial x_i}, \quad \ddot{y}_i + 2\dot{y}_i = \frac{\partial U}{\partial y_i} \quad and \quad \ddot{z}_i = \frac{\partial U}{\partial z_i}, \quad (i = 1, 2). \tag{3}$$

Equilibrium points of the system are those points where –

$$\dot{x}_i = \dot{y}_i = \dot{z}_i = \frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial y_i} = \frac{\partial U}{\partial z_i} = 0, \text{ (i = 1 , 2).} \tag{4}$$

Garain (2013) found four values of L_4 in which two values (x_i, y_i, 0), (i = 1, 2) are known as in- line triangular equilibrium points and other two values (x_i, y_i, 0), (i = 1, 2) are known as perpendicular triangular equilibrium points of L_4.

3. CHARACTERISTIC EXPONENT FOR THE STABILITY OF L_4

Similar to Whipple (1984), the Characteristic equation corresponding to the point L_4 (x_1, y_1, 0) is

$$f(\lambda) = \lambda^4 + \lambda^2\left[4 - \frac{1}{\mu_1}(U_{x_1x_1} + U_{y_1y_1})\right] + \frac{1}{\mu_1^2}\left[U_{x_1x_1}U_{y_1y_1} - (U_{x_1y_1})^2\right] = 0. \tag{5}$$

Let, $x_1 = P_{11} + \varepsilon Q_{11}, x_2 = P_{21} + \varepsilon Q_{21}, y_1 = R_{11} + \varepsilon S_{11}, y_2 = R_{21} + \varepsilon S_{21}$ (6)

Where, $P_{11}, Q_{11}, R_{11}, S_{11}, P_{21}, Q_{21}, R_{21}$ *and* S_{21} are known values obtained from Garain (2013).

Let, $r_{11}^{-3} = a_1 + \varepsilon b_1$ (7)

Where,

$$a_1 = (P_{11}^2 + R_{11}^2 - 2P_{11}\mu + \mu^2)^{-\frac{3}{2}}, b_1 = -3(P_{11}Q_{11} - Q_{11}\mu + R_{11}S_{11})$$
$$(P_{11}^2 + R_{11}^2 - 2P_{11}\mu + \mu^2)^{-\frac{5}{2}}. \tag{8}$$

$$r_{21}^{-3} = a_2 + \varepsilon b_2 \tag{9}$$

Where,

$$a_2 = \{P_{11}^2 + R_{11}^2 - 2P_{11}(1-\mu) + (1-\mu)^2\}^{-\frac{3}{2}}, b_2 = -3\{P_{11}Q_{11} - Q_{11}(1-\mu) + R_{11}S_{11})$$
$$\{P_{11}^2 + R_{11}^2 - 2P_{11}(1-\mu) + (1-\mu)^2\}^{-\frac{5}{2}}. \tag{10}$$

$$r^{-3} = a_3 + \varepsilon b_3 \tag{11}$$

Where,

$$a_3 = \{(P_{11} - P_{21})^2 + (R_{11} - R_{21})^2\}^{-\frac{3}{2}}, b_3 = -3\{(P_{11} - P_{21})(Q_{11} - Q_{21})$$
$$+ (R_{11} - R_{21})(S_{11} - S_{21})\}\{(P_{11} - P_{21})^2 + (R_{11} - R_{21})^2\}^{-\frac{5}{2}}. \tag{12}$$

$$r_{11}^{-5} = a_4 + \varepsilon b_4 \tag{13}$$

Where,

$$a_4 = (P_{11}^2 + R_{11}^2 - 2P_{11}\mu + \mu^2)^{-\frac{5}{2}}, b_1 = -5(P_{11}Q_{11} - Q_{11}\mu + R_{11}S_{11})$$
$$(P_{11}^2 + R_{11}^2 - 2P_{11}\mu + \mu^2)^{-\frac{7}{2}}.$$

(14)

$$r_{21}^{-5}=a_5+\varepsilon b_5$$ (15)

Where,

$$a_5=\{P_{11}^2+R_{11}^2-2P_{11}(1-\mu)+(1-\mu)^2\}^{-\frac{5}{2}}, b_5=-5\{P_{11}Q_{11}-Q_{11}(1-\mu)+R_{11}S_{11}\}\{P_{11}^2+R_{11}^2-2P_{11}(1-\mu)+(1-\mu)^2\}^{-\frac{7}{2}}.$$

(16)

$$r^{-5}=a_6+\varepsilon b_6$$ (17)

Where,

$$a_6=\{(P_{11}-P_{21})^2+(R_{11}-R_{21})^2\}^{-\frac{5}{2}}, b_6=-5\{(P_{11}-P_{21})(Q_{11}-Q_{21})+(R_{11}-R_{21})(S_{11}-S_{21})\}\{(P_{11}-P_{21})^2+(R_{11}-R_{21})^2\}^{-\frac{7}{2}}.$$

(18)

$$U_{x_1x_1}=u_{41}+\varepsilon v_{41}$$ (19)

Where,

$$u_{41}=\mu_1[1-a_1(1-\mu)-\mu a_2-\mu_2 a_3+3a_4(1-\mu)(P_{11}^2+\mu^2-2P_{11}\mu)+3\mu a_5(P_{11}-\mu+1)^2+3\mu a_6(P_{11}-P_{21})^2],$$

(20)

$$v_{41}=\mu_1[(1-\mu)(b_1-a_1)-\mu b_2-\mu_2 b_3+6a_4Q_{11}(1-\mu)(\mu-P_{11})+3(P_{11}^2+\mu^2-2P_{11}\mu)(1-\mu)(b_4-a_4)+6a_6\mu_2(P_{11}-P_{21})(Q_{11}-Q_{21})+3\mu_2 b_6(P_{11}-P_{21})^2].$$

(21)

$$U_{x_1y_1}=u_{42}+\varepsilon v_{42}$$ (22)

Where,

$$u_{42}=\mu_1[3(1-\mu)(a_4P_{11}R_{11}-\mu a_4R_{11})+3\mu(P_{11}-\mu+1)a_5R_{11}+3\mu_2 a_6(P_{11}-P_{21})(R_{11}-R_{21})],$$

(23)

$$v_{42}=\mu_1[3(1-\mu)\{a_4Q_{11}R_{11}+a_4P_{11}S_{11}-a_4\mu S_{11}+(b_4-a_4)(P_{11}R_{11}-\mu R_1)\}+3\mu(P_1-\mu+1)(a_5S_{11}+b_5R_{11}+a_5Q_{11}R_{11})+3\mu_2\{a_6(P_{11}-P_{21})(S_{11}-S_{21})+a_6(Q_{11}-Q_{21})(R_{11}-R_{21})+b_6(P_{11}-P_{21})(R_{11}-R_{21})\}].$$

(24)

$$U_{y_1y_1}=u_{43}+\varepsilon v_{43}$$ (25)

Where,

$$u_{43}=\mu_1[1-(1-\mu)a_1-\mu a_2-\mu_2 a_3+3(1-\mu)a_4R_{11}^2+3\mu a_5R_{11}^2+3\mu_2 a_6(R_{11}-R_{21})^2],$$

(26)

$$v_{43} = \mu_1[(1-\mu)(b_1 - a_1) - \mu b_2 - \mu_2 b_3 - 6(1-\mu) a_4 R_{11} S_{11} + 3(1-\mu) R_{11}^2 (b_4 - a_4)$$
$$+ 6\mu a_5 R_{11} S_{11} + 3\mu b_5 R_{11}^2 + 6\mu_2 a_6 (R_{11} - R_{21})(S_{11} - S_{21}) + 3\mu_2 b_6 (R_{11} - R_{21})^2].$$
(27)

The Characteristic equation f(λ) = 0 reduces to

$$f(\lambda) = \lambda^4 + A_{41}\lambda^2 + B_{41} = 0 \tag{28}$$

Where, $A_{41} = 4 - \frac{1}{\mu_1}(U_{x_1x_1} + U_{y_1y_1}) = a_{41} + \varepsilon b_{41}$ (29)

Here, $a_{41} = 4 - \frac{1}{\mu_1}(u_{41} + u_{43}), b_{41} = \frac{1}{\mu_1}(v_{41} + v_{43}).$ (30)

$$B_{41} = \frac{1}{\mu_1^2}\{U_{x_1x_1}U_{y_1y_1} - (U_{x_1y_1})^2\} = a_{42} + \varepsilon b_{42} \tag{31}$$

Where, $a_{42} = \frac{1}{\mu_1^2}(u_{41}u_{43} - u_{42}^2), b_{42} = \frac{1}{\mu_1^2}(u_{41}v_{43} + u_{43}v_{41} - 2u_{42}v_{42})$ (32)

Since, the equation (28) is quadratic in λ^2 and let, D be its discriminant, then

$$D = A_{41}^2 - 4B_{41} = C_{41} + D_{41} \tag{33}$$

Where, $C_{41} = a_{41}^2 - 4a_{42}$ *and* $D_{41} = 2a_{41}b_{41} - 4b_{42}$. (34)

Let, two roots of λ^2 in equation (28) be e_{41} and e_{42} respectively.

Where, $e_{41} = f_{41} + \varepsilon g_{41}$, (35)

$$f_{41} = \frac{1}{2}[\frac{1}{\mu_1}(u_{41} + u_{43}) - 4 + \sqrt{C_{41}}], g_{41} = \frac{1}{2}[\frac{1}{\mu_1}(v_{41} + v_{43}) + \frac{D_{41}}{2\sqrt{C_{41}}}] \tag{36}$$

$$e_{42} = f_{42} + \varepsilon g_{42}, \tag{37}$$

$$f_{42} = \frac{1}{2}[\frac{1}{\mu_1}(u_{41} + u_{43}) - 4 - \sqrt{C_{41}}], g_{42} = \frac{1}{2}[\frac{1}{\mu_1}(v_{41} + v_{43}) - \frac{D_{41}}{2\sqrt{C_{41}}}]. \tag{38}$$

We may get the value of λ^2 for given values of μ and μ_i (i =1, 2).

Table-1
Values of square of Characteristic roots
($\mu_1 = \mu_2 = 10^{-12}$, $e_{41} = f_{41} + \varepsilon\ g_{41}$, $e_{42} = f_{42} + \varepsilon\ g_{42}$)

μ	f_{41}	g_{41}	f_{42}	g_{42}
0.00001	-1312238.02	-2549619861755210000000000	-1316824.14	2549619861755210000000000
0.00002	-2625818.26	-1802813866542750000000000	-2632304.01	1802813866542750000000000
0.00003	-3939618.78	-1471959006860670000000000	-3947562.17	1471959006860670000000000
0.00004	-5253532.98	-1274725878485190000000000	-5262705.22	1274725878485190000000000
0.00005	-6567519.58	-1140124430977810000000000	-6577774.45	1140124430977810000000000
0.00006	-7881557.38	-1040763577900670000000000	-7892791.03	1040763577900670000000000
0.00007	-9195633.82	-963538666096512000000000	-9207767.54	963538666096512000000000
0.00008	-10509740.69	-901288087497031000000000	-10522712.18	901288087497031000000000
0.00009	-11823872.32	-849723886806562000000000	-11837630.65	849723886806562000000000
0.0001	-13138024.52	-806101138284399000000000	-13152527.09	806101138284399000000000
0.0002	-26280224.78	-569874381834469000000000	-26300734.47	569874381834469000000000
0.0003	-39423052.20	-465198319402587000000000	-26300734.47	569874381834469000000000
0.0004	-52566169.68	-402785136347623000000000	-52595174.69	402785136347623000000000
0.0005	-65709446.63	-360182934585379000000000	-65741875.18	360182934585379000000000
0.0006	-78852816.08	-328728414846246000000000	-78888339.72	328728414846246000000000
0.0007	-91996238.25	-304276648790630000000000	-92034608.12	304276648790630000000000
0.0008	-105129687.20	-284562363051238000000000	-105180706.25	284562363051238000000000
0.0009	-118283144.91	-268229189919302000000000	-118326652.13	268229189919302000000000
0.001	-131426598.27	-254408794434280000000000	-131472458.85	254408794434280000000000
0.002	-262859452.80	-179501380523058000000000	-262924308.56	179501380523058000000000
0.003	-394287332.53	-146243397379887000000000	-394366763.20	146243397379887000000000
0.004	-525709162.68	-126375939376083000000000	-525800880.07	126375939376083000000000
0.005	-657124521.58	-112789908287239000000000	-657227063.33	112789908287239000000000
0.006	-788533188.70	-102741001786707000000000	-788645516.02	102741001786707000000000
0.007	-919935029.48	-94915650564926500000000	-920056355.20	94915650564926500000000
0.008	-1051329953.23	-88595675029946600000000	-1051459654.10	88595675029946600000000
0.009	-1182717894.28	-83350866268803700000000	-1182855460.94	83350866268803700000000
0.01	-1314098802.46	-78905610982404800000000	-1314243808.46	78905610982404800000000
0.02	-2627514614.47	-54641916142043200000000	-2627719655.61	54641916142043200000000
0.03	-3940203269.81	-43719648302812300000000	-3940454357.98	43719648302812300000000
0.04	-5252152790.03	-37120691093269600000000	-5252442680.92	37120691093269600000000
0.05	-6563353331.05	-32563157446541800000000	-6563677393.12	32563157446541800000000
0.06	-7873795754.20	-29160748857938800000000	-7874150695.96	29160748857938800000000
0.07	-9183471262.23	-26486404199348600000000	-9183854587.85	26486404199348600000000
0.08	-10492371265.4	-24304842202545700000000	-10492780998.5	24304842202545700000000
0.09	-11800487323.5	-22474088308223700000000	-11800921847.6	22474088308223700000000
0.1	-13107811114.6	-20902213783289300000000	-13108269075.9	20902213783289300000000
0.2	-26135708142.4	-11400763474037500000000	-26136354807.9	11400763474037500000000
0.3	-39075845354.7	-5463392863622220000000	-39076636063.7	5463392863622220000000

It is clear from Table (1) that e_{41} is negative and the sign of e_{42} depends on the value of ε.

For, $e_{42} < 0$, we must have $\varepsilon < \frac{|f_{42}|}{g_{42}}$.

4. CONCLUSION

In Table-1 we get that for $\varepsilon < 5.16479 \times 10^{-18}$, the value of λ^2 is negative. So the value of each of the characteristic root will be purely imaginary and in this case L_4 is stable.

References

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