Non-existence of $[z-\frac{1}{\sqrt{2}}(t_1+t_2)]$-type and $[t_1+t_2)/z\sqrt{2}]$-type plane wave solutions in V_4 with cosmological term λ

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Abstract

The plane wave solutions g_{ij} can not satisfy the Einstein's field equations $R_{ij}-\frac{1}{2}Rg_{ij}=\lambda g_{ij}$ with the cosmological constant for $[z-\frac{1}{\sqrt{2}}(t_1+t_2)]$-type and $[(t_1+t_2)/z\sqrt{2}]$ -type plane waves in V_4 where λ is a cosmological constant. Furthermore this study has been extended to the most general solutions $(g_{ij,}F_{ij})$ where gravitational field coexists with the electromagnetic field.

1. Introduction : Takeno (1961) has solved field equations in general theory of relativity and obtained the plane wave solutions g_{ij} of the field equations $R_{ij}=0$ in the forms of $(z-t)$-type and (t/z)-type plane waves in purely gravitational case with the proper choice of the co-ordinate systems. Furthermore he has obtained the most general solutions $(g_{ij,}F_{ij})$ of the field equations in general relativity for $(z-t)$-type and (t/z)-type plane waves in the case where gravitational field coexists with the electromagnetic field.

On the lines of Takeno (1961), in the paper [1], we have obtained the plane wave solutions g_{ij} of the field equations $R_{ij}=0$ in purely gravitational case in the form of

$$P=\frac{\overline{\overline{m}}}{2m}-\frac{\overline{m}^2}{4m^2}-\frac{\overline{m}\,\overline{B}}{2mB}+\frac{\overline{\overline{B}}}{2B}-\frac{3\overline{B}^2}{4B^2}=0 \quad (1.1)$$

and established the existence of $[z-\frac{1}{\sqrt{2}}(t_1+t_2)]$-type and $[(t_1+t_2)/z\sqrt{2}]$-type plane waves using the plane symmetric space-time

$$ds^2=-Ady^2-2Bdz^2+2Bdt_1^2+2Bdt_2^2 \quad \text{and} \quad (1.2)$$

$$ds^2=-Ady^2-2Z^2Bdz^2+2Bdt_1^2+2Bdt_2^2 \quad (1.3)$$

respectively by reformulating Takeno's (1961) definition as follows:

Definition A plane wave g_{ij} is a non-flat solution of field equations

$$R_{ij} = 0, \qquad (i, j = 1,2,3,4) \tag{1.4}$$

in an empty region of space-time such that

$$g_{ij} = g_{ij}(Z), \qquad Z = Z(x^i), \qquad \text{where } x^i = y, z, t_1, t_2 \tag{1.5}$$

in some suitable co-ordinate system such that

$$g^{ij} Z,_i Z,_j = 0, \qquad Z,_i = \frac{\partial Z}{\partial x^i}, \tag{1.6}$$

$$Z = Z(z, t_1, t_2), \qquad Z,_2 \neq 0, \qquad Z,_3 \neq 0, \qquad Z,_4 \neq 0. \tag{1.7}$$

In this definition, the signature convention adopted is

$$g_{rr} < 0, \quad r = 1,2$$

$$\begin{vmatrix} g_{rr} & g_{rs} \\ g_{sr} & g_{ss} \end{vmatrix} > 0, \qquad g_{33} > 0, \qquad g_{44} > 0 \tag{1.8}$$

[not summed for r , s =1,2] and accordingly $g = \det(g_{ij}) > 0$. (1.9)

Furthermore, we have investigated the most general solutions $(g_{ij,} F_{ij})$ of the field equations in general relativity for $[z - \frac{1}{\sqrt{2}}(t_1 + t_2)]$-type and $[(t_1 + t_2)/z\sqrt{2}]$-type plane waves in the case of gravitational field coexists with the electromagnetic field where g_{ij} are given by (1.2) and (1.3) and F_{ij} are given by

$$F_{12} = \sigma, \qquad F_{13} = -\sigma/\sqrt{2}, \qquad F_{14} = -\sigma/\sqrt{2}, \qquad F_{34} = 0 \tag{1.10}$$

and $$F_{12} = \sigma, \qquad F_{13} = -(\frac{z}{t_1 + t_2})\sigma, \qquad F_{14} = -(\frac{z}{t_1 + t_2})\sigma, \qquad F_{34} = 0 \tag{1.11}$$

respectively.

In the present paper, we examine the existence of these plane wave solutions for the field equations

$$R_{ij} - \frac{1}{2} R g_{ij} = \lambda g_{ij}, \quad (i, j = 1,2,3,4) \tag{1.12}$$

and $$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi E_{ij} + \lambda g_{ij}, \qquad (i, j = 1,2,3,4) \tag{1.13}$$

with the introduction of cosmological term in the case of purely gravitational field as well as where the gravitational field coexists with the electromagnetic field.
We study these plane wave solutions in two cases.

Case I . Purely gravitational field

2. $[z-\frac{1}{\sqrt{2}}(t_1+t_2)]$**-type plane wave in** V_4

From (1.2) we have

$$g_{11}=-A,\ g_{22}=-2B,\ g_{33}=2B. \tag{2.1}$$

Then components of Ricci tensors are related as under

$$R_{11}=0,\ \frac{M^2}{2}R_{22}=M^2R_{33}=M^2R_{44}=-\frac{M^2}{\sqrt{2}}R_{23}=-\frac{M^2}{\sqrt{2}}R_{24}=M^2R_{34}=P \tag{2.2}$$

where $P=\frac{\bar{\bar{m}}}{2m}-\frac{\bar{m}^2}{4m^2}-\frac{\bar{m}\bar{B}}{2mB}+\frac{\bar{\bar{B}}}{2B}-\frac{3\bar{B}^2}{4B^2}$. (2.3)

Also

$$R=g^{ij}R_{ij}=\frac{1}{2B}[-\frac{2P}{M^2}+\frac{P}{M^2}+\frac{P}{M^2}]=0 \quad \text{where} \quad M=-\sqrt{2}. \tag{2.4}$$

Hence field equation (1.12) takes the form

$$R_{ij}=\lambda g_{ij}. \tag{2.5}$$

For $i=1,\ j=1$ the equation (2.5) yields

$$R_{11}=\lambda g_{11}=\lambda(-A)$$

$$\Rightarrow -A\lambda=0, \qquad \because R_{11}=0.$$

$$\Rightarrow \lambda=0 \quad \text{which is contradiction} \qquad (\because A\neq 0) \tag{2.6}$$

Thus from the equation (2.6) we have
$[z-\frac{1}{\sqrt{2}}(t_1+t_2)]$-*type plane wave in the form of* g_{ij} *can not satisfy the field equations* (1.12) *with the cosmological constant.*

3. $[(t_1+t_2)/z\sqrt{2}]$**-type plane wave in** V_4

From (1.3) we have

$$g_{11}=-A\,,\; g_{22}=-2Z^2B\,, g_{33}=2B\,, g_{44}=2B\,. \tag{3.1}$$

Then we have

$$R_{11}=0\,, R_{22}=2Z^2R_{33}=2Z^2R_{44}=-Z\sqrt{2}R_{23}=-Z\sqrt{2}R_{24}=2Z^2R_{34}=P'Z^2 \tag{3.2}$$

where $P'=\frac{1}{z^2}[\frac{\bar{\bar{m}}}{2m}-\frac{\bar{m}^2}{4m^2}-\frac{\bar{m}\,\bar{B}}{2mB}+\frac{\bar{\bar{B}}}{2B}-\frac{3\bar{B}^2}{4B^2}]\,.$ (3.3)

Also $R=g^{ij}R_{ij}=\frac{1}{2B}[-P'+P']=0\,.$ (3.4)

Hence field equation (1.12) again becomes

$$R_{ij}=\lambda g_{ij} \qquad (i\,,\, j=1,2,3,4) \tag{3.5}$$

For $i=1,\; j=1$ the equation (3.3) yields

$$R_{11}=\lambda g_{11}=\lambda(-A)$$

$$\Rightarrow -A\lambda=0\,, \qquad\qquad \because R_{11}=0\,.$$

$$\Rightarrow \lambda=0 \qquad \text{which is contradiction} \qquad (\because A\neq 0) \tag{3.6}$$

Thus from the equation (3.6) we have

$[(t_1+t_2)/z\sqrt{2}]$*-type plane wave in the form of* g_{ij} *can not satisfy the field equation* (1.12) *with the cosmological constant.*

Case II Gravitational field coexists with electromagnetic field

4. $[z-\frac{1}{\sqrt{2}}(t_1+t_2)]$**-type plane wave in** V_4

From (1.2) and (1.10) we have

$$g_{11}=-A\,, \qquad g_{22}=-2B\,, \qquad g_{33}=2B\,, \qquad g_{44}=2B\,, \tag{4.1}$$

and $F_{12} = \sigma$, $F_{13} = -\frac{\sigma}{\sqrt{2}}$, $F_{14} = -\frac{\sigma}{\sqrt{2}}$, $F_{34} = 0$. (4.2)

Then we have

$$R_{11} = 0, \frac{M^2}{2} R_{22} = M^2 R_{33} = M^2 R_{44} = -\frac{M^2}{\sqrt{2}} R_{23} = -\frac{M^2}{\sqrt{2}} R_{24} = M^2 R_{34} = P, \quad (4.3)$$

$$E_{22} = -2E_{33} = 2E_{44} = -\sqrt{2}E_{23} = -\sqrt{2}E_{24} = 2E_{34} = -\frac{\sigma^2}{m} = -\frac{P}{8\pi} \quad (4.4)$$

where $P = \frac{8\pi\sigma^2}{m}$. (4.5)

Also $R = g^{ij} R_{ij} = \frac{1}{2B}[-\frac{2P}{M^2} + \frac{P}{M^2} + \frac{P}{M^2}] = 0$ where $M = -\sqrt{2}$ (4.6)

Hence equation (1.13) becomes

$$R_{ij} = -8\pi E_{ij} + \lambda g_{ij} \quad (i, j = 1, 2, 3, 4) \quad (4.7)$$

$$\Rightarrow R_{ij} = -8\pi[\frac{1}{4} g_{ij} F_{kl} F^{kl} - F_{ik} F_{jl} g^{kl}] + \lambda g_{ij}$$

$$\Rightarrow R_{ij} = -8\pi[-(F_{ik} F_{jl} g^{kl})] + \lambda g_{ij}, \quad \because F_{kl} F^{kl} = 0. \quad (4.8)$$

For $i = 1$, $j = 1$ we have

$$R_{11} = -8\pi[-(F_{1k} F_{1l} g^{kl})] + \lambda g_{11} = -8\pi[-\frac{\sigma^2}{2B} + \frac{\sigma^2}{2B}] + \lambda(-A) = \lambda(-A)$$

$$\Rightarrow -A\lambda = 0, \quad \because R_{11} = 0$$

$\Rightarrow \lambda = 0$ which is contradiction $(\because A \neq 0)$. (4.9)

For $i = 2$, $j = 2$

$$R_{22} = -8\pi[-(F_{2k} F_{2l} g^{kl})] + \lambda g_{22}$$

$$\Rightarrow \lambda B = 0, \quad \because B \neq 0$$

$\Rightarrow \lambda = 0$ which is contradiction. (4.10)

For $i = 3, \; j = 3$

$$R_{33} = -8\pi[-(F_{3k}F_{3l}g^{kl})] + \lambda g_{33}$$

$\Rightarrow \lambda B = 0$, $\because B \neq 0$

$\Rightarrow \lambda = 0$ which is contradiction. (4.11)

For $i = 4, \; j = 4$

$$R_{44} = -8\pi[-(F_{4k}F_{4l}g^{kl})] + \lambda g_{44}$$

$\Rightarrow \lambda B = 0$, $\because B \neq 0$

$\Rightarrow \lambda = 0$ which is contradiction. (4.12)

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For $i = 2, \; j = 3$

$$R_{23} = -8\pi[-(F_{2k}F_{3l}g^{kl})] + \lambda g_{23}$$

$\Rightarrow 0 = 0$. Thus equation is satisfied. (4.13)

For $i = 2 \; \; j = 4$

$$R_{24} = -8\pi[-(F_{2k}F_{4l}g^{kl})] + \lambda g_{24}$$

$\Rightarrow 0 = 0$. Thus equation is satisfied. (4.14)

For $i = 3, \; j = 4$

$$R_{34} = -8\pi[-(F_{3k}F_{4l}g^{kl})] + \lambda g_{34}$$

$\Rightarrow 0 = 0$. Thus equation is satisfied. (4.15)

Thus from the equations (4.9), (4.10), (4.11) and (4.12) we have $[z - \frac{1}{\sqrt{2}}(t_1 + t_2)]$-*type plane wave in the form of* $(g_{ij,} F_{ij})$ *can not satisfy the field equation* (1.13) *with the cosmological constant.*

5. $[(t_1+t_2)/z\sqrt{2}]$ **-type plane wave in** V_4

From (1.3) and (1.11) we have

$$g_{11}=-A,\quad g_{22}=-2Z^2B,\quad g_{33}=2B,\quad g_{44}=2B\ . \tag{5.1}$$

and $F_{12}=\sigma, F_{13}=-(\frac{z}{t_1+t_2})\sigma=-\frac{\sigma}{Z\sqrt{2}}, F_{14}=-(\frac{z}{t_1+t_2})\sigma=-\frac{\sigma}{Z\sqrt{2}}, F_{34}=0.$ (5.2)

Then we have

$$R_{11}=0, R_{22}=2Z^2R_{33}=2Z^2R_{44}=-Z\sqrt{2}R_{23}=-Z\sqrt{2}R_{24}=2Z^2R_{34}=P'Z^2, \tag{5.3}$$

$$E_{22}=2Z^2E_{33}=2Z^2E_{44}=-Z\sqrt{2}E_{23}=-Z\sqrt{2}E_{24}=2Z^2E_{34}=-\frac{\sigma^2}{m}=-\frac{P'Z^2}{8\pi} \tag{5.4}$$

where $P^{'}=\frac{8\pi\sigma^2}{mZ^2}$. (5.5)

Also $R=g^{ij}R_{ij}=\frac{1}{2B}[-P'+P']=0$. (5.6)

Hence equation (1.13) becomes

$$R_{ij}=-8\pi E_{ij}+\lambda g_{ij} \tag{5.7}$$

$$\Rightarrow R_{ij}=-8\pi[\frac{1}{4}g_{ij}F_{kl}F^{kl}-F_{ik}F_{jl}g^{kl}]+\lambda g_{ij} \tag{5.8}$$

For $i=1,\ j=1$

$$R_{11}=-8\pi[-(F_{1k}F_{1l}g^{kl})]+\lambda g_{11}$$

$\Rightarrow -A\lambda=0,$.

$\Rightarrow \lambda=0$ which is contradiction. (5.9)

For $i=2,\ j=2$

$R_{22}=-8\pi[-(F_{2k}F_{2l}g^{kl})]+\lambda g_{22} \quad \Rightarrow -\lambda Z^2B=0$

$\Rightarrow \lambda=0$ which is contradiction. (5.10)

For $i=3,\ j=3$

$R_{33}=-8\pi[-(F_{3k}F_{3l}g^{kl})]+\lambda g_{33}\qquad \Rightarrow \lambda B=0,\qquad \because B\neq 0$

$\Rightarrow \lambda=0$ which is contradiction. (5.11)

For $i=4,\ j=4$

$R_{44}=-8\pi[-(F_{4k}F_{4l}g^{kl})]+\lambda g_{44}\qquad \Rightarrow \lambda B=0,\qquad \because B\neq 0$

$\Rightarrow \lambda=0$ which is contradiction. (5.12)

For $i=2,\ j=3$

$R_{23}=-8\pi[-(F_{2k}F_{3l}g^{kl})]+\lambda g_{23}$

$\Rightarrow 0=0$. Thus equation is satisfied. (5.13)

For $i=2,\ j=4$

$R_{24}=-8\pi[-(F_{2k}F_{4l}g^{kl})]+\lambda g_{24}$

$\Rightarrow 0=0$. Thus equation is satisfied. (5.14)

For $i=3,\ j=4$

$R_{34}=-8\pi[-(F_{3k}F_{4l}g^{kl})]+\lambda g_{34}$

$\Rightarrow 0=0$. Thus equation is satisfied. (5.15)

Thus from the equations (5.9), (5.10), (5.11) and (5.12) we have $[(t_1+t_2)/z\sqrt{2}]$*-type plane wave in the form of* $(g_{ij,}F_{ij})$ *can not satisfy the field equation* (1.13) *with the cosmological constant.*

Conclusion *The equations* (2.6) *&* (3.6) *and* (4.9),(4.10),(4.11),(4.12) *&* (5.9),(5.10),(5.11), (5.12) *show that there does not exists any plane wave solution* g_{ij} *and* $(g_{ij,}F_{ij})$ *of the field equations* (1.12) *and* (1.13) *respectively with the introduction of cosmological term* λ *for* $[z-\frac{1}{\sqrt{2}}(t_1+t_2)]$*-type and* $[(t_1+t_2)/z_2]$*-type plane waves.*

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