On two theorems of Katre-Rao-Sheth

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Abstract

In [2] Katre-Rao-Seth gave an explicit solution of the system of linear equations $E\varphi E^t = \varphi'$, where φ, φ' are alternating matrices of Pfaffian 1, which at most differ in their first row and column, and E is of the form $\prod_{i=2}^{2r} E_{1i}(x_i)$. To prove this they proved two theorems viz. Theorem 2.4 and Theorem 3.6. In this paper, we show that these two theorems are equivalent.

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1. Introduction

Let φ,φ' be alternating matrices of Pfaffian one. In ([2], Theorem 2.4) an explicit formula is given for the x_i, $2 \le i \le 2m$, where $E = \prod_{j=2}^{2m} E_{1j}(x_j)$, for solving the problem $E\varphi E^t = \varphi$'. They show that

Theorem 1.1([2], Theorem 2.4): *For* $m \ge 2$, $x^t = Bv^{*t}$ *is a solution of the equation* $E\varphi E^t = \varphi$'.

In the same paper, in ([2], Theorem 3.6) it is shown:

Theorem 1.2*Let* $v, v' \in Um_{r+1}(R)$, $r \ge 1$, *and* $\langle v, w\rangle = 1 = \langle v', w\rangle$. *Then there is a sequence of pairs, starting with* $(v_0, w_0) = (v, w)$, *and ending with* $(v_n, w_n) = (v', w)$, *such that, for* $i \ge 0$, *the pairs* (v_{i+1}, w_{i+1}) *has* v_{i+1} *as a Cohn transform of* v_i *w.r.t. w, and* $w_{i+1} = w$, $(v, w) = (v_0, w_0) \to (v_1, w_1) \to \cdots \to (v_n, w_n) = (v', w)$.

The key lemma used to prove this is:

Lemma 1.3 Let $v, v', w \in R^{r+1}$, $r \ge 1$,such that $\langle v, w\rangle = 1 = \langle v', w\rangle$. Let $v = (a_0, \dots, a_r)$, $v' = (a_0', \dots, a_r')$, $w = (b_0, \dots, b_r)$. Then $a_k' - a_k = \sum_{i\neq k}(a_k' a_i - a_k a_i')b_i$.

In this paper we show, by an example, that these two theorems are intimately related.

2. Preliminaries

Let R be a commutative ring with 1.

Definition 2.1*A matrix* $A \in M_n(R)$ *is said to be alternating if* $a_{ii} = 0$ *and* $a_{ij} = -a_{ji}$, *for* $1 \le i, j \le n$. *The space of all alternating* $n \times n$ *matrices over a commutative ring R will be denoted by* $Alt_n(R)$. *It is clearly a free R-module of rank* $1 + 2 + \cdots + (n-1) = \binom{n}{2}$ *with* *basis* $B_{ij} = e_{ij} - e_{ji}$, $1 \leq i < j \leq n$, *where* $e_{ij} \in M_n(R)$ *with ij-th entry is 1 and all other entries are 0.*

Definition 2.2 (Pfaffian of a skew-symmetric matrix A)
Pfaffian of a skew-symmetric matrix A is defined, upto sign, as the polynomial in the entries of A whose square is $\det(A)$ *and is denoted by pf(A).*

Definition 2.3 *The **General Linear** group* $GL_r(R)$ *is defined as the group of* $r \times r$ *invertible matriceswith entries in R.*

Definition 2.4 *The **Special Linear** group is denoted by* $SL_r(R)$ *and is defined as* $SL_r(R) = \{\alpha \in GL_r(R) : \det(\alpha) = 1\}$.

Definition 2.5 *Thegroup of **elementary matrices**$E_r(R)$ isa subgroup of$GL_r(R)$ generated by matrices of the form* $E_{ij}(\lambda) = I_r + \lambda e_{ij}$, *where* $\lambda \in R$, $i \neq j$ *and* $e_{ij} \in M_r(R)$ *with ij-th entry is*1 *and allother entries are* 0.

3. On Two Theorems of Katre-Rao-Seth

In this section, by using Lemma 1.3 we show that we can give an explicit procedure to get Theorem 1.1. This procedure can also be used in higher sizes. We detailed this calculation in Proposition 3.2.

Proposition 3.1 Let φ_1, φ_2 be two 6×6 alternating matrices with Pfaffian 1 over R, with $(\varphi_1)_{ij} = (\varphi_2)_{ij}$where $2 \leq i, j \leq 6$. Then there exists $E = \begin{pmatrix} 1 & x \\ 0 & I_5 \end{pmatrix}$,where $x \in M_{15}(R)$, such that $E\, \varphi_1 E^t = \varphi_2$.

Proof: We give the proof in a procedural way. Let

$$\varphi_1 = \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ -x_1 & 0 & y_2 & y_3 & y_4 & y_5 \\ -x_2 & -y_2 & 0 & z_3 & z_4 & z_5 \\ -x_3 & -y_3 & -z_3 & 0 & w_4 & w_5 \\ -x_4 & -y_4 & -z_4 & -w_4 & 0 & u_5 \\ -x_5 & -y_5 & -z_5 & -w_5 & -u_5 & 0 \end{pmatrix} \text{and} \varphi_2 = \begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ -a_1 & 0 & y_2 & y_3 & y_4 & y_5 \\ -a_2 & -y_2 & 0 & z_3 & z_4 & z_5 \\ -a_3 & -y_3 & -z_3 & 0 & w_4 & w_5 \\ -a_4 & -y_4 & -z_4 & -w_4 & 0 & u_5 \\ -a_5 & -y_5 & -z_5 & -w_5 & -u_5 & 0 \end{pmatrix}$$

Let $v_1 = e_1\varphi_1 = (x_1, x_2, x_3, x_4, x_5)$and $v_2 = e_1\varphi_2 = (a_1, a_2, a_3, a_4, a_5)$.If $w = e_1\varphi_1^* = (b_1, b_2, b_3, b_4, b_5)$, where φ_1^* denote the transpose inverse of φ_1, then $\langle v_1, w \rangle = 1 = \langle v_2, w \rangle$. Note that

$$b_1 = u_5z_3 + w_4z_5 - w_5z_4, \quad b_2 = -u_5y_3 - w_4y_5 + w_5y_4,$$
$$b_3 = u_5y_2 + y_4z_5 - y_5z_4, b_4 = -w_5y_2 + y_3z_5 - y_5z_3,$$

$$b_5 = w_4y_2 - y_3z_4 - y_4z_3$$

Now as $\langle v_1, w \rangle = 1 = \langle v_2, w \rangle$, one get $\sum_{i=1}^{5} x_i b_i = 1 = \sum_i^5 a_i b_i$, so $\sum_{i=1}^{5}(x_i - a_i)b_i = 0$.Hence,

$$a_k - x_k = (a_k - x_k)\sum_{i=1}^{5} x_i b_i = (a_k - x_k)x_k b_k + (a_k - x_k)\sum_{i\neq k} x_i b_i$$
$$= \sum_{i\neq k}(x_i - a_i)b_i x_k + (a_k - x_k)\sum_{i\neq k} x_i b_i = \sum_{i\neq k}(a_k x_i - x_k a_i)b_i .$$

Thus we have $a_k = x_k + \sum_{i\neq k}(a_k x_i - x_k a_i)b_i$ and henceone can write the value of a_k explicitly as

$$a_1 = x_1 - y_2(x_1a_3u_5 - a_1x_3u_5 + x_1a_5w_4 - x_1a_4w_5 + a_1x_4w_5 - a_1x_5w_4)$$
$$- y_3(a_1x_2u_5 - x_1a_2u_5 - a_1x_4z_5 + x_1a_4z_5 + a_1x_5z_4 - x_1a_5z_4) - y_4(x_1a_2w_5$$
$$-a_1x_2w_5 + a_1x_3z_5 - x_1a_5z_3 - x_1a_3z_5 - a_1x_5z_3) - y_5(a_1x_2w_4 - x_1a_2w_4$$
$$+ x_1a_3z_4 - a_1x_3z_4 + a_1x_4z_3 - x_1a_4z_3)$$

$$a_2 = x_2 + y_2(a_2x_3u_5 - x_2a_3u_5 + x_2a_4w_5 - a_2x_4w_5 + a_2x_5w_4 - x_2a_5w_4)$$
$$- z_3(x_2a_1u_5 - a_2x_1u_5 + a_2x_4y_5 - x_2a_4y_5 + x_2a_5y_4 - a_2x_5z_4)$$
$$- z_4(a_2x_1w_5 - x_2a_1w_5 + x_2a_3y_5 - a_2x_3y_5 + a_2x_5y_3$$
$$- x_2a_5y_3) - z_5(x_2a_1w_4 - a_2x_1w_4 + a_2x_3y_4 - x_2a_3y_4 + a_2x_4y_3$$
$$+ x_2a_4y_3)$$

$$a_3 = x_3 + y_3(a_2x_3u_5 - x_2a_3u_5 + a_3x_4z_5 - x_3a_4z_5 + x_3a_5z_4 - a_3x_5z_4)$$
$$+ z_3(a_3x_1u_5 - a_1x_3u_5 + a_4x_3y_5 - x_4a_3y_5 + x_5a_3y_4 - a_5x_3y_4)$$
$$- w_4(a_1x_3z_5 - x_1a_3z_5 + x_2a_3y_5 - a_2x_3y_5 + a_5x_3y_2 - x_5a_3y_2)$$
$$- w_5(x_1a_3z_4 - a_1x_3z_4 + a_2x_3y_4 - x_2a_3y_4 + a_3 x_4y_2 - x_3a_4y_2)$$

$$a_4 = x_4 + y_4(a_4x_2w_5 - x_4a_2w_5 + x_4a_3z_5 - a_4x_3z_5 + a_4x_5z_3 - x_4a_5z_3)$$
$$+ z_4(x_4a_1w_5 - a_4x_1w_5 + a_4x_3y_5 - x_4a_3y_5 + x_4a_5y_3 - a_4x_5y_3)$$
$$+ w_4(a_4x_1z_5 - x_4a_1z_5 + x_4a_2y_5 - a_4x_2y_5 + a_4x_5y_2 - x_4a_5y_2)$$
$$- u_5(x_4a_1z_3 - a_4x_1z_3 + a_4x_2y_3 - x_4a_2y_3 - a_3x_4y_2 + x_3a_4y_2)$$

$$a_5 = x_5 + y_5(a_2x_5w_4 - x_5a_2w_4 + x_3a_5z_4 - a_3x_5z_4 + a_5x_4z_3 - x_5a_4z_3)$$
$$+ z_5(x_1a_5w_4 - a_1x_5w_4 + a_3x_5y_4 - x_3a_5y_4 + x_4a_5y_3 - a_4x_5y_3)$$
$$+ w_5(a_1x_5z_4 - x_1a_5z_4 + x_2a_5y_4 - a_2x_5y_4 + a_5x_4y_2 - x_5a_4y_2)$$
$$- u_5(x_1a_5z_3 - a_1x_5z_3 + a_2x_5y_3 - x_2a_5y_3 + a_5x_3y_2 - x_5a_3y_2)$$

For the following computation, we use the mathematical software MuPAD. Note that for

$$\lambda_1 = x_1a_3u_5 - a_1x_3u_5 + x_1a_5w_4 - x_1a_4w_5 + a_1x_4w_5 - a_1x_5w_4$$
$$\lambda_2 = a_1x_2u_5 - x_1a_2u_5 - a_1x_4z_5 + x_1a_4z_5 + a_1x_5z_4 - x_1a_5z_4$$
$$\lambda_3 = x_1a_2w_5 - a_1x_2w_5 + a_1x_3z_5 - x_1a_5z_3 - x_1a_3z_5 - a_1x_5z_3$$
$$\lambda_4 = a_1x_2w_4 - x_1a_2w_4 + x_1a_3z_4 - a_1x_3z_4 + a_1x_4z_3 - x_1a_4z_3$$

one has $a_1 = x_1 - y_2\lambda_2 - y_3\lambda_2 - y_4\lambda_3 - y_5\lambda_4$.

Hence we consider $E_{16}(\lambda_4)E_{15}(\lambda_3)E_{14}(\lambda_2)E_{13}(\lambda_1)\varphi_1\big(E_{16}(\lambda_4)E_{15}(\lambda_3)E_{14}(\lambda_2)E_{13}(\lambda_1)\big)^t$, which is same as

$$\alpha_1 = \begin{pmatrix} 0 & a_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \\ -a_1 & 0 & y_2 & y_3 & y_4 & y_5 \\ -\mu_2 & -y_2 & 0 & z_3 & z_4 & z_5 \\ -\mu_3 & -y_3 & -z_3 & 0 & w_4 & w_5 \\ -\mu_4 & -y_4 & -z_4 & -w_4 & 0 & u_5 \\ -\mu_5 & -y_5 & -z_5 & -w_5 & -u_5 & 0 \end{pmatrix}.$$

where

$\mu_2 = x_2 + a_2u_5x_3y_2 - a_3u_5x_2y_2 - a_1u_5x_2z_3 + a_2u_5x_1z_3 + a_2w_4x_5y_2 - a_2w_5x_4y_2 + a_4w_5x_2y_2 - a_5w_4x_2y_2 - a_1w_4x_2z_5 + a_1w_5x_2z_4 + a_2w_4x_1z_5 - a_2w_5x_1z_4.$

$\mu_3 = x_3 + a_2u_5x_3y_3 - a_3u_5x_2y_3 - a_1u_5x_3z_3 + a_3u_5x_1z_3 + a_2w_4x_5y_3 - a_2w_5x_4y_3 + a_4w_5x_2y_3 - a_5w_4x_2y_3 - a_1w_4x_3z_5 + a_1w_5x_3z_4 + a_3w_4x_1z\_5 - a_3w_5x_1z\_4$

$\mu_4 = x_4 + a_2u_5x_3y_4 - a_3u_5x_2y_4 - a_1u_5x_4z_3 + a_4u_5x_1z_3 + a_2w_4x_5y_4 - a_2w_5x_4y_4 + a_4w_5x_2y_4 - a_5w_4x_2y_4 - a_1w_4x_4z_5 + a_1w_5x_4z_4 + a_4w_4x_1z_5 - a_4w_5x_1z_4$

$\mu_5 = x_5 + a_2u_5x_3y_5 - a_3u_5x_2y_5 - a_1u_5x_5z_3 + a_5u_5x_1z_3 + a_2w_4x_5y_5 - a_2w_5x_4y_5 + a_4w_5x_2y_5 - a_5w_4x_2y_5 - a_1w_4x_5z_5 + a_1w_5x_5z_4 + a_5w_4x_1z_5 - a_5w_5x_1z_4$

Now we can write

$$a_2 - \mu_2 = -z_3\lambda_5 - z_4\lambda_6 - z_5\lambda_7,$$

where

$\lambda_5 = a_2x_4y_5 - x_2a_4y_5 + x_2a_5y_4 - a_2x_5y_4$, $\lambda_6 = x_2a_3y_5 - a_2x_3y_5 + a_2x_5y_3 - x_2a_5y_3$,
$\lambda_7 = a_2x_3y_4 - x_2a_3y_4 - a_2x_4y_3 + x_2a_4y_3$.

Hence we consider $E_{16}(\lambda_7)E_{15}(\lambda_6)E_{14}(\lambda_5)\alpha_1\big(E_{16}(\lambda_8)E_{15}(\lambda_7)E_{14}(\lambda_6)\big)^t$ which is same as

$$\alpha_2 = \begin{pmatrix} 0 & a_1 & a_2 & \mu_6 & \mu_7 & \mu_8 \\ -a_1 & 0 & y_2 & y_3 & y_4 & y_5 \\ -a_2 & -y_2 & 0 & z_3 & z_4 & z_5 \\ -\mu_6 & -y_3 & -z_3 & 0 & w_4 & w_5 \\ -\mu_7 & -y_4 & -z_4 & -w_4 & 0 & u_5 \\ -\mu_8 & -y_5 & -z_5 & -w_5 & -u_5 & 0 \end{pmatrix} \text{where}$$

$\mu_6 = x_3 + a_2u_5x_3y_3 - a_3u_5x_2y_3 - a_1u_5x_3z_3 + a_3u_5x_1z_3 + a_2w_4x_3y_5 - a_2w_5x_3y_4 - a_3w_4x_2y_5 + a_3w_5x_2y_4 - a_1w_4x_3z_5 + a_1w_5x_3z_4 + a_3w_4x_1z_5 - a_3w_5x_1z_4$

$\mu_7 = x_4 + a_2u_5x_4y_3 - a_4u_5x_2y_3 - a_1u_5x_4z_3 + a_4u_5x_1z_3 + a_2w_4x_4y_5 - a_2w_5x_4y_4 - a_4w_4x_2y_5 + a_4w_5x_2y_4 - a_1w_4x_4z_5 + a_1w_5x_4z_4 + a_4w_4x_1z_5 - a_4w_5x_1z_4$

$\mu_8 = x_5 + a_2u_5x_5y_3 - a_5u_5x_2y_3 - a_1u_5x_5z_3 + a_5u_5x_1z_3 + a_2w_4x_5y_5 - a_2w_5x_5y_4 - a_5w_4x_2y_5 + a_5w_5x_2y_4 - a_1w_4x_5z_5 + a_1w_5x_5z_4 + a_5w_4x_1z_5 - a_5w_5x_1z_4$

We now write $a_3 - \mu_6 = y_3\lambda_8 + z_3\lambda_9 - w_4\lambda_{10} - w_5\lambda_{11}$, where

$\lambda_8 = a_3x_4z_5 - x_3a_4z_5 + x_3a_5z_4 - a_3x_5z_4$, $\lambda_9 = x_3a_4y_5 - a_3x_4y_5 + a_3x_5y_4 - x_3a_5y_4$
$\lambda_{10} = x_3a_5y_2 - a_3x_5y_2$, $\lambda_{11} = a_3x_4y_2 - x_3a_4y_2$

Then $E_{16}(\lambda_{11})E_{15}(\lambda_{10})E_{13}(\lambda_9)E_{12}(\lambda_8)\alpha_2\big(E_{16}(\lambda_{11})E_{15}(\lambda_{11})E_{13}(\lambda_9)E_{12}(\lambda_8)\big)^t$ becomes

$$\alpha_3 = \begin{pmatrix} 0 & a_1 & a_2 & a_3 & \mu_9 & \mu_{10} \\ -a_1 & 0 & y_2 & y_3 & y_4 & y_5 \\ -a_2 & -y_2 & 0 & z_3 & z_4 & z_5 \\ -a_3 & -y_3 & -z_3 & 0 & w_4 & w_5 \\ -\mu_9 & -y_4 & -z_4 & -w_4 & 0 & u_5 \\ -\mu_{10} & -y_5 & -z_5 & -w_5 & -u_5 & 0 \end{pmatrix}$$

where

$\mu_9 = x_4 + a_2u_5x_4y_3 - a_3u_5x_4y_2 - a_4u_5x_2y_3 + a_4u_5x_3y_2 - a_1u_5x_4z_3 + \mathrm{a}_4\mathrm{u}_5\mathrm{x}_1\mathrm{z}_3$
$+ a_2w_4x_4y_5 - a_2w_5x_4y_4 - a_4w_4x_2y_5 + a_4w_5x_2y_4 - a_1w_4x_4z_5 + a_1w_5x_4z_4$
$+ a_4w_4x_1z_5 - a_4w_5x_1z_4 + a_3x_4y_4z_5 - a_3x_4y_5z_4 - a_4x_3y_4z_5 + a_4x_3y_5z_4$
$\mu_{10} = x_5 + a_2u_5x_5y_3 - a_3u_5x_5y_2 - a_5u_5x_2y_3 + a_5u_5x_3y_2 - a_1u_5x_5z_3 + a_5u_5x_1z_3$
$+ a_2w_4x_5y_5 - a_2w_5x_5y_4 - a_5w_4x_2y_5 + a_5w_5x_2y_4 - a_1w_4x_5z_5 + a_1w_5x_5z_4$
$+ a_5w_4x_1z_5 - a_5w_5x_1z_4 + a_3x_5y_4z_5 - a_3x_5y_5z_4 - a_5x_3y_4z_5 + a_5x_3y_5z_4$

Again we write $a_4 - \mu_9 = y_4\lambda_{12} + z_4\lambda_{13} + w_4\lambda_{14}$,where
$\lambda_{12} = a_4x_5z_3 - a_5x_4z_3, \quad \lambda_{13} = a_5x_4y_3 - a_4x_5y_3, \lambda_{14} = a_4x_5y_2 - a_5x_4y_2$

Then $E_{14}(\lambda_{14})E_{13}(\lambda_{13})E_{12}(\lambda_{12})\alpha_3\left(E_{14}(\lambda_{14})E_{13}(\lambda_{13})E_{12}(\lambda_{12})\right)^t$becomes

$$\alpha_4 = \begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ -a_1 & 0 & y_2 & y_3 & y_4 & y_5 \\ -a_2 & -y_2 & 0 & z_3 & z_4 & z_5 \\ -a_3 & -y_3 & -z_3 & 0 & w_4 & w_5 \\ -a_4 & -y_4 & -z_4 & -w_4 & 0 & u_5 \\ -a_5 & -y_5 & -z_5 & -w_5 & -u_5 & 0 \end{pmatrix} = \varphi_2$$

Thus if
$\varepsilon_1 = E_{16}(\lambda_4)E_{15}(\lambda_3)E_{14}(\lambda_2)E_{13}(\lambda_1), \quad \varepsilon_2 = E_{16}(\lambda_7)E_{15}(\lambda_6)E_{14}(\lambda_5),$
$\varepsilon_3 = E_{16}(\lambda_{11})E_{15}(\lambda_{10})E_{13}(\lambda_9)E_{12}(\lambda_8), \quad \varepsilon_4 = E_{14}(\lambda_{14})E_{13}(\lambda_{13})E_{12}(\lambda_{12})$
and $E = \varepsilon_4\varepsilon_3\varepsilon_2\varepsilon_1$, then E is of the form $\begin{pmatrix} 1 & x \\ 0 & I_5 \end{pmatrix}$ and $E\varphi_1E^t = \varphi_2$, where $x = (\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 \quad \beta_5)$.

$\beta_1 = a_2u_5x_3 - a_3u_5x_2 + a_2w_4x_5 - a_2w_5x_4 + a_4w_5x_2 - a_5w_4x_2 + a_3x_4z_5 - \mathrm{a}_3\mathrm{x}_5\mathrm{z}_4$
$- \mathrm{a}_4\mathrm{x}_3\mathrm{z}_5 + a_4x_5z_3 + a_5x_3z_4 - a_5x_4z_3$
$\beta_2 = - a_1u_5x_3 + a_3u_5x_1 - a_1w_4x_5 + a_1w_5x_4 - a_4w_5x_1 + a_5w_4x_1 - a_3x_4y_5$
$+ a_3x_5y_4 + a_4x_3y_5 - a_4x_5y_3 - a_5x_3y_4 + a_5x_4y_3$
$\beta_3 = a_1u_5x_2 - a_2u_5x_1 + a_2x_4y_5 - a_2x_5y_4 - a_4x_2y_5 + a_4x_5y_2 + a_5x_2y_4$
$- a_5x_4y_2 - a_1x_4z_5 + a_1x_5z_4 + a_4x_1z_5 - a_5x_1z_4$
$\beta_4 = - a_1w_5x_2 + a_2w_5x_1 - a_2x_3y_5 + a_2x_5y_3 + a_3x_2y_5 - a_3x_5y_2 - a_5x_2y_3$
$+ a_5x_3y_2 + a_1x_3z_5 - a_1x_5z_3 - a_3x_1z_5 + a_5x_1z_3$
$\beta_5 = a_1w_4x_2 - a_2w_4x_1 + a_2x_3y_4 - a_2x_4y_3 - a_3x_2y_4 + a_3x_4y_2 + a_4x_2y_3$
$- a_4x_3y_2 - a_1x_3z_4 + a_1x_4z_3 + a_3x_1z_4 - a_4x_1z_3$
This $x = (\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 \quad \beta_5)$can also calculated in the following way. Note that one can write

$$\varphi^* = \begin{pmatrix} 0 & w \\ -w^t & B \end{pmatrix}$$

where$B$ is a 5×5 submatrix of φ^*. Then $Bv_2^t = (\theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \quad \theta_5)$where
$\theta_1 = a_2u_5x_3 - a_3u_5x_2 + a_2w_4x_5 - a_2w_5x_4 + a_4w_5x_2 - a_5w_4x_2 + a_3x_4z_5$
$- a_3x_5z_4 - a_4x_3z_5 + a_4x_5z_3 + a_5x_3z_4 - a_5x_4z_3$
$\theta_2 = - a_1u_5x_3 + a_3u_5x_1 - a_1w_4x_5 + a_1w_5x_4 - a_4w_5x_1 + a_5w_4x_1 - a_3x_4y_5$
$+ a_3x_5y_4 + a_4x_3y_5 - a_4x_5y_3 - a_5x_3y_4 + a_5x_4y_3$
$\theta_3 = a_1u_5x_2 - a_2u_5x_1 + a_2x_4y_5 - a_2x_5y_4 - a_4x_2y_5 + a_4x_5y_2 + a_5x_2y_4$
$- a_5x_4y_2 - a_1x_4z_5 + a_1x_5z_4 + a_4x_1z_5 - a_5x_1z_4$
$\theta_4 = - a_1w_5x_2 + a_2w_5x_1 - a_2x_3y_5 + a_2x_5y_3 + a_3x_2y_5 - a_3x_5y_2 - a_5x_2y_3$
$+ a_5x_3y_2 + a_1x_3z_5 - a_1x_5z_3 - a_3x_1z_5 + a_5x_1z_3$

$\theta_5 = a_1w_4x_2 - a_2w_4x_1 + a_2x_3y_4 - a_2x_4y_3 - a_3x_2y_4 + a_3x_4y_2 + a_4x_2y_3 - a_4x_3y_2 - a_1x_3z_4 + a_1x_4z_3 + a_3x_1z_4 - a_4x_1z_3$
Thus $Bv_2^t = x^t$ as required.

We can extend the Proposition 3.1to any *n* as follows:

Theorem 3.2*Let φ_1, φ_2 be two $n \times n$ alternating matrices with Pfaffian 1 over R, with $(\varphi_1)_{ij} = (\varphi_2)_{ij}$ where $2 \leq i,j \leq n$. Then there exists $E = \begin{pmatrix} 1 & x \\ 0 & I_{n-1} \end{pmatrix}$, where $x \in M_{1,n-1}(R)$, such that $E\,\varphi_1 E^t = \varphi_2$.*

Proof: Let $v_1 = e_1\varphi_1 = (x_1, x_2, \dots, x_{n-1})$ and $v_2 = e_1\varphi_2 = (a_1, a_2, \dots, a_{n-1})$. If $w = e_1\varphi_1^* = (b_1, b_2, \dots, b_{n-1})$, where φ_1^* denote the transpose inverse of φ_1, then $\langle v_1, w \rangle = 1 = \langle v_2, w \rangle$. Thus by Lemma 1.3, $a_k = x_k + \sum_{i \neq k}(a_kx_i - x_ka_i)b_i$. Note that one can write

$$a_1 = x_1 + \varphi_{32}\lambda_1 + \varphi_{42}\lambda_2 + \cdots + \varphi_{n2}\lambda_{n-1}$$

for some $\lambda_1, \lambda_2, \dots, \lambda_{n-1} \in R$ .Let $\varepsilon_1 = E_{1n}(\lambda_{n-2})E_{1,n-1}(\lambda_{n-3}) \cdots E_{14}(\lambda_2)E_{13}(\lambda_1)$ and $\alpha_1 = \varepsilon_1\,\varphi\,\varepsilon_1^t$.Then $e_1\alpha_1 = (a_1, \mu_1, \mu_2, \dots, \mu_{n-1})$ and $(\alpha_1)_{ij} = (\varphi_1)_{ij}$ where $2 \leq i,j \leq n$ for some $\mu_1, \mu_2, \dots, \mu_{n-1} \in R$. Now we write

$$a_2 - \mu_2 = \varphi_{43}\delta_1 + \varphi_{53}\delta_2 + \cdots + \varphi_{(n-2)3}\delta_{n-2}$$

for some $\delta_1, \delta_2, \dots, \delta_{n-2} \in R$. Let

$$\varepsilon_2 = E_{1n}(\delta_{n-2})E_{1,n-1}(\delta_{n-3}) \cdots E_{15}(\delta_2)E_{14}(\delta_1) \text{and } \alpha_2 = \varepsilon_2\alpha_1\varepsilon_2^t.$$

Then $e_1\alpha_2 = (a_1, a_2, \nu_1, \dots, \nu_{n-2})$ and $(\alpha_2)_{ij} = (\varphi_1)_{ij}$ where $2 \leq i,j \leq n$ for some $\nu_1, \nu_2, \dots, \nu_{n-2} \in R$. Now we write $a_3 - \nu_1 = \varphi_{54}\theta_1 + \varphi_{64}\theta_2 + \cdots + \varphi_{n3}\theta_{n-3}$ for some $\theta_1, \theta_2, \dots, \theta_{n-3} \in R$ .Let $\varepsilon_3 = E_{1n}(\theta_{n-3})E_{1(n-1)}(\theta_{n-4}) \cdots E_{14}(\theta_1)$ and $\alpha_3 = \varepsilon_3\alpha_2\varepsilon_3^t$. Then $e_1\alpha_3 = (a_1, a_2, a_3, k_1 \dots, k_{n-3})$ and $(\alpha_3)_{ij} = (\varphi_1)_{ij}$ where $2 \leq i,j \leq n$ for some $k_1, k_2, \dots, k_{n-3} \in R$.

Proceeding in this way, one gets, for some ε_{n-2}, $\varepsilon_{n-2}\alpha_{n-3}\varepsilon_{n-2}^t = \varphi_2$. Let $E = \varepsilon_{n-2}\varepsilon_{n-3} \dots \varepsilon_2\varepsilon_1$. We also write $\varphi^* = \begin{pmatrix} 0 & w \\ -w^t & B \end{pmatrix}$where B is a$n-1 \times n-1$ submatrix of φ^*. Then E is of the form$\begin{pmatrix} 1 & x \\ 0 & I_{n-1} \end{pmatrix}$and $E\varphi_1E^t = \varphi_2$, where $x = (\beta_1 \quad \beta_2 \quad \cdots \quad \beta_{n-1}) = Bv_2^t$.

Remark 3.3 Thus we have proved that Theorem 1.2 implies Theorem 1.1.

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