Anti-magic labeling of Triangular Trees

Manisha Acharya[a] , Ganesh Joshi[b]
[a] Department of Mathematics, Maharshi Dayanand College, Parel, Mumbai 400012, MS, India
[b] Department of Mathematics, Maharshi Dayanand College, Parel, Mumbai 400012, MS, India

Abstract

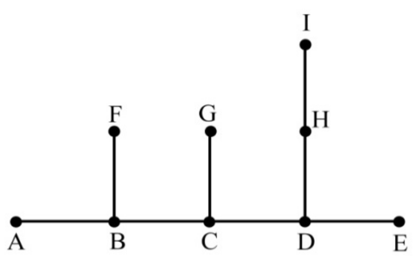
An anti-magic labeling of a finite simple undirected graph G is a bijection from the set of edges E(G) to the set of integers {1, 2,..., |E(G)|} such that the vertex sums are pairwise distinct, where the vertex sum at one vertex is the sum of labels of all edges incident to such vertex. A graph is called anti-magic if it admits an anti-magic labeling. In this paper, we established an anti-magic labeling of Triangular Tree Graphs.

KEYWORDS: Anti-magic, Anti-magic labeling, Trees, Special trees.

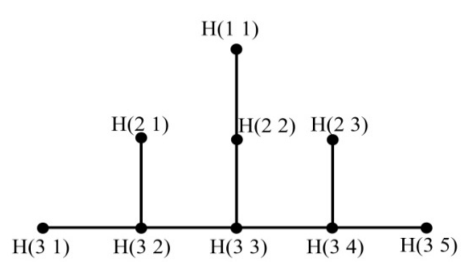
1. Introduction

In this paper we consider only simple undirected finite graphs with V (G) to be the set of vertices of the graph G and E(G) to be the set of edges of the graph G. Let us give definition of an anti-magic graph.

Definition 1.1. A bijective function f : E(G) → {1, 2,..., |E(G)|} is said to be labeling of graph G. The vertex sum function associated with a labeling f of graph G is a function S : V (G) → N defined as S(u) = $\sum_{v \in N(u)} f(uv)$. That is, S(u) is the sum of labels of the edges incident with u. A labeling f of a graph G is said to be anti-magic if the vertex sum function associated with f is an injective function. A graph having an anti-magic labeling is said to be anti-magic. Hartsfield and Ringel [3] conjectured that all connected graphs except K_2 are anti-magic. An excellent survey on graph labeling can be found in [2]. As a step towards proving this conjecture, Miller, M. Phanalasy, O. Ryan, J., & Rylands, L. J. [5] provided a method whereby, given any degree sequence pertaining to a tree, we can construct an anti-magic tree based on this sequence. Furthermore, swapping the roles of edges and vertices with respect to a labeling, they provided a method to construct an edge anti-magic vertex labeling for any tree. For the given degree sequence there can be more than 1 non isomorphic graph. For eg.



Tree(a)



Tree(b)

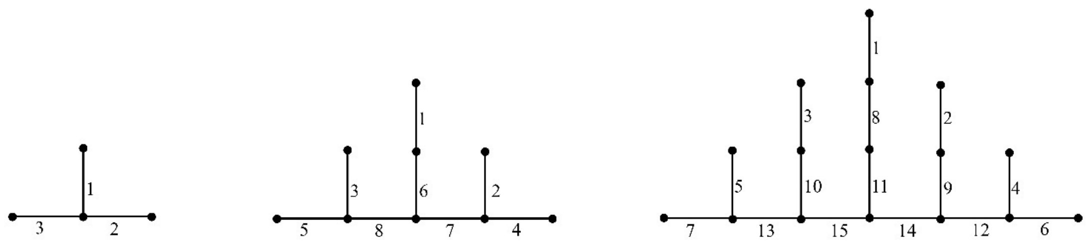
Both the trees shown in the above diagram have the same degree sequence 1,1,1,1,1,2,3,3,3 but the trees are not isomorphic as the longest path of Tree(a) has 5 edges and that for Tree(b) has only 4 edges. We call Tree(b) as the triangular tree of height 3. At the height 1 there is only one vertex we named it as H(1 1). At the height 2 there are 3 vertices and we named them as H(2 1), H(2 2), H(2 3) from left to right. At the height 3 there are 5 vertices we named them as H(31), H(3 2), . . . , H(3 5) from left to right. We define the triangular tree of height n>1 denoted by TT(n) to be generalized graph of Tree(b)[1]. Simple counting gives order of the tree n^2 and the size of the tree n^2-1.

2. Triangular Tree

In this section we will prove that triangular trees are anti-magic. To avoid tedious notations we use some shortcuts of writing. for eg H(i j)H(i+1 j+1) = 7 means an edge H(i j)H(i+1 j+1) is labeled as 7, SH(i j) = 10 we mean the vertex sum associated with the vertex H(i j) is 10.

Theorem 1: TT(n) is anti-magic for n<5.

Proof:



TT(2) **TT(3)** **TT(4)**

The edge labeling for TT(2) shows SH(1 1) = 1, SH(2 1) = 3, SH(2 2) = 6, SH(2 3) = 2 which all are distinct.

The edge labeling of TT(3) shows SH(1 1) = 1, SH(2 1) = 3, SH(2 2) = 7, SH(2 3) = 2, SH(3 1) = 5, SH(3 2) = 16, SH(3 3) = 21, SH(3 4) = 13, SH(3 5) = 4 which all are distinct.

The edge labeling of TT(4) shows SH(11) = 1, SH(2 1) = 3, SH(2 2) = 9, SH(2 3) = 2, SH(3 1) = 5, SH(3 2) = 13, SH(3 3) = 19, SH(3 4) = 11, SH(3 5) = 4, SH(4 1) = 7, SH(4 2) = 25, SH(4 3) = 38, SH(4 4) = 40, SH(4 5) = 35, SH(4 6) = 22, SH(4 7) = 6 which all are distinct. Therefore TT(n) is anti-magic for n<5 ■

Theorem 2: For all even numbers n>4, TT(n) is anti-magic.

Poof: Label the edges as follows.

H(11)H(2 2) = 1

For j=1 to n-1,

H(j 1)H(j+1 2) = 2j-1, H(j 2j-1)H(j+1 2j) = 2(j-1) [Note these are the labels of pendant edges]

For j=2 to n-2 and for i=1 to (n-1)-j,

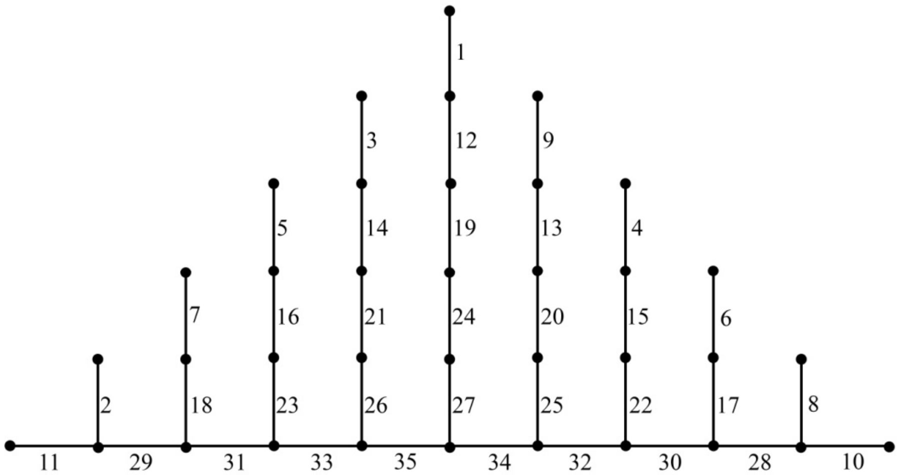
H(j j)H(j+1 j+1) = $2n(j-1)-j^2+4$
H(j+i j)H(j+i+1 j+1) = $2n(j-1)-j^2+4+2i$
H(j+i j+2i)H(j+i+1 j+2i+1) = $2n(j-1)-j^2+4+2i-1$
H(n-1 n-1)H(n n) = $2n(n-3)-(n-2)^2+7$
H(n 1)H(n 2) = 2(n-1)+1
H(n 2n-2)H(n 2n-1) = 2(n-1)
For k=2 to (n-1)
H(n k)H(n k+1) = 2n(n-2)-(n-1)2+4+2(k-1)
H(n 2n-(k+1))H(n 2n-k) = 2n(n-2)-(n-1)2+4+2(k-1)-1
Swap the labels of H(2 3)H(3 4) and H(n-1 1)H(n 2).
We claim that the above labeling is anti-magic.
It can be observed that for the given combinations of i, j, k the set of edge of labels is { 1,2,3, . . . , $2n^2$-n-1}.Thus the mapping is injective with E(TT(n).
The edge labels of the pendant edges are 1, 2, . . . , 2n-1. Thus the associated sum at pendant vertices are all distinct and less than 2n. The degrees of all other vertices are either 2 or 3 and at every such vertex there is at least one edge incident with it having its edge label greater than 2n-1. Therefore the associated sum at any pendant vertex is less than the associated sum at any non pendant vertex. Hence the associated sum at any pendant vertex is not equal to the associated sum at any non pendant vertex. Here onwards we will not consider the associated sum at pendant vertices. Let H(x y) be any vertex of degree 2. We will prove that SH(x y) is odd except for SH(3 4)
If 2< y ≤x, SH(x y) = H(x-1 y-1)H(x y)+H(x y)H(x+1 y+1) = H((y-1)+((x-1)-(y-1)) y-1)H(x y)+H(y+(x-y) y)H(y+(x-y)+1 y+1) = $[2n(y-2)-(y-1)^2+4+2((x-1)-(y-1))]+[2n(y-1)-y^2+4-2(x-y)]=2n(2y-3)+8+2(2x-2y)-(2y^2-2y+1)$ which is odd.
If x< y < 2x-2, SH(x y) = H(x-1 y-1)H(x y)+H(x y)H(x+1 y+1)=H((x-1)-((y-1)-(x-1))+((y-1)-(x-1)) (x-1)-((y-1)-(x-1))+2((y-1)-(x-1)))H(x y)+ H(x-(y-x)+(y-x) x-(y-x)+2(y-x))H(x+1 y+1) = $[2n((x-1)-((y-1)-(x-1)))-((x-1)-((y-1)-(x-1)))^2+4+2((y-1)-(x-1))-1]+[2n(x-(y-x))-(x-(y-x))^2+4+2(y-x)-1]=2n(4x-2y-1)+4(y-x)+2(2x-y)^2-2(2x-y)+7$ which is odd.
If y=2, SH(x 2) = H(x-1 1)H(x 2)+H(x 2)H(x+1 3) = [2(x-1)-1]+[2n+2(x-2)]=2n+4x-7 which is odd
If y=2x-2 such that (x y) ≠ (3 4), SH(x 2x-2) = H(x-1 2x-2-1)H(x 2x-2)+H(x 2x-2)H(x+1 2x-2+1) = [2(x-2)]+[2n+2(x-2)-1] = 2n+4x-9 which is odd. Consider SH(3 4) = H(2 3)H(3 4)+H(3 4)H(4 5) = (2n-3)+(2n+1) = 4n-2 which is even. Hence among the vertices of degree 2 the only even sum associated with these vertices is SH(3 4).Now we will calculate the sums associated with the vertices at height n. SH(n 2) =H(n 1)H(n 2)+H(n 2)H(n 3)+H(n-1 1)H(n 2) = $[2n-1]+[2n(n-2)-(n-1)^2+4+2]+[2] = 3n^2-4n+6$ which is even.SH(n 2n-2) = H(n 2n-2)H(n 2n-1)+H(n 2n-3)H(n 2n-2)+H(n-1 2n-3)H(n 2n-2) = $[2n-2]+[2n(n-2)-(n-1)^2+4+1]+[2(n-1-1)] = n^2+2n-2$ which is even.
For 3≤ k ≤n-1, SH(n k) = H(n k)H(n k+1)+H(n k-1)H(n k)+H(n-1 k-1)H(n k) = $[2n(n-2)-(n-1)^2+4+2(k-1)]+[2n(n-2)-(n-1)^2+4+2(k-2)]+[2n(k-2)-(k-1)^2+4+2((n-1)-(k-1))]$ = $2(n-1)(n-2)+(2n+2)k-(k-1)^2$. Similarly SH(n 2n-k) = $2(n-1)(n-2)+(2n+2)k-(k-1)^2-3$ and SH(n n) = $3n^2-2n$. We can see that $\frac{d}{dk}$SH(n k) = 2(n+2-k) > 0 hence SH(n k) is increasing function with k. Therefore with this fact and with the simple comparison the sequence of even sums associated with the vertices of height n can be written as (and it is obvious SH(3 4) < SH(n 2)) SH(n 2) < SH(n 2n-2) < SH(n 3) < SH(n 5) <……..<

SH(n n-1) < SH(n n) and (it is obvious SH(3 4)< SH(n 2) with) SH(n 2) < SH(n 2n-2) < SH(n 2n-4) < SH(n 2n-6) < ……..< SH(n n+2) < SH(n n). Hence SH(3 4), SH(n 2), SH(n 2n-2) mutually differ. They also differ from the even sums associated at the base vertices. They naturally differ from the odd sums associated at the base vertices and sum associated with the vertices of degree 2. Hence for the further comparisons of associated sums we will not consider SH(3 4), SH(n 2), SH(n 2n-2). Now we will prove that the associated sums with the degree 2 vertices mutually differ. For j= 2 to n-2,We call the sets{H(j j)},{H(j+1 j), H(j+2 j), ….., H(j+(n-1-j) j)},{H(j+1 j+2), H(j+2 j+4), …..,H(j+(n-1-j) j+2(n-1-j))}as j-top, j-left and j-right vertices respectively and the union of these sets we call as the j-triangle vertices. For every j-top, from the labeling it can be seen that labels of downward edges starting from every j-right vertices are increasing with i for i=1 to n-1-j. Similarly it can be seen that labels of downward edges starting from every j-left vertices are increasing with i for i =1 to n-1-j. For every j-top the downward edge starting from it has smaller label than the downward edge starting from H(j+1 j+2). For any fix j height where $3 \leq j \leq n-2$, H(j+i j+2i)H(j+i+1 j+2i+1) < H(j+i j)H(j+i+1 j+1).Therefore H(j j)H(j+1 j+1) < H(j+1 j+2)H(j+2 j+3) < H(j+1 j)H(j+2 j+1) < H(j+2 j+4)H(j+3 j+5) < H(j+2 j)H(j+3 j+1) <………< H(n-1 2(n-1)-j)H(n 2(n-1)-j+1) < H(n-1 j)H(n j+1). (The argument similarly works for j=2) By replacing j with j-1 in the above inequality and adding these inequalities we get SH(j j) < SH(j+1 j+2) < SH(j+1 j) < SH(j+2 j+4) < SH(j+2 j) <………< SH(n-1 2(n-1)-j) < SH(n-1 j). It can be seen that H(1 1)H(2 2) < H(2 1)H(3 2) < H(3 5)H(4 6) < H(3 1)H(4 2) <………< H(n-1 2(n-1)-j)H(n 2(n-1)-j+1) < H(n-1 j)H(n j+1). Hence we get SH(2 2) < SH(3 2) < SH(4 6) < SH(4 2) <………< SH(n-1 2(n-1)-2) < SH(n-1 2) [SH(3 4) is of opposite parity of these vertices is already discussed] Hence we can conclude sum associated with vertices of j- triangle mutually differ. For any height j where $3 \leq j \leq n-2$ we also have SH(j j) = H(j-1 j-1)H(j j)+H(j j)H(j+1 j+1) = $[2n(j-2)-(j-1)^2+4]+[2n(j-1)-(j)^2+4] = 4nj-6n-2j^2+2j+7$ which is the lowest associated sums among the j- triangle vertices. SH(n-1 j)=H(n-2 j-1)H(n-1 j)+H(n-1 j)H(n j+1) = $[2n(j-2)-(j-1)^2+4+2(n-1-j)]+[2n(j-1)-(j)^2+4+2(n-1-j)] = 4nj-2n-2j-2j^2+3$ which is the highest associated sums among the j- triangle vertices. For 2-triangle vertices the lowest associated sum is SH(2 2) = 2n+1 and the highest associated sum is SH(n-1 2) = 6n-11 Let H(x y) be any point lying below the (imaginary) lines of {H(j j), H(j+1 j), H(j+2 j), ….., H(j+(n-1-j) j)} and{ H(j j)H(j+1 j+2), H(j+2 j+4), …..,H(j+(n-1-j) j+2(n-1-j))}Therefore x > j and either ($y \leq x$ implies H(x y) lies on y- triangle and y > j) or (y > x implies H(x y) lies on (x-(y-x))- triangle and it can be noted x-(y-x) > j). If $y \leq x$ then SH(y y)-SH(n-1 j) = $[4ny-6n-2y^2+2y+7]-[4nj-2n-2j-2j^2+3] = (y-j-1)(4n-2y-2j)+4>0$ as $j < n$ and $y \leq n$. Therefore SH(n-1 j) < SH(y y) < SH(x y). If y>x implies H(x y) lies on (x-(y-x))-triangle and x-(y-x) > j then SH(y y)-SH(n-1 j) = (2x-y-j-1)(4n-2(2x-y)-2j)+4 > 0 as $j < n$ and $2x-y \leq n$. Therefore SH(n-1 j) < SH(2x-y 2x-y) < SH(x y).Therefore for any H(x y) lying in j-triangle we get the SH(x y) is greater than sum associated with j-triangle vertices. The same can be seen for 2- triangle too. As H(n-1 n-1) lies in every j- triangle hence SH(n-1 n-1) > Sum associated with any j-triangle vertex, which shows sums associated with vertices of degree 2 mutually differ. Sums associated with degree 2 vertices are odd hence naturally differ from even Sums associated with height n vertices. From the earlier discussions and expressions obtained for SH(n k), SH(n 2n-k), SH(n n) we get the lowest odd sum associated with non-pendant vertices at height n is SH(n 2n-

3) = 2(n-1)(n-2)+3(2n+2)-$(3-1)^2$-3 = $2n^2$+3. The highest odd sum among the vertices of degree 2 is SH(n-1 n-1) = $2n^2$-4n+3, hence SH(n-1 n-1)< SH(n 2n-3). Thus the sums associated with vertices of degree 2 differ from every odd sums associated with the vertices at the height n. Now it is only left to prove the sums associated with the vertices at height n mutually differ. From the earlier discussion it is clear that SH(n 2), SH(n 2n-2) differ mutually and they also differ from the sum associated with all other vertices at height n. As discussed earlier SH(n n) is greater than all the sums associated with the vertices of height n. Consider SH(n t) where 3≤ t ≤ 2n-3 and t≠n. It is readily known that SH(n t) is increasing for 3≤ t ≤ n-1 and SH(n t) is decreasing for n+1≤ t ≤ 2n-3.Therefore no two vertices on the left of H(n n) have the associated sums equal and no two vertices on the right of H(n n) have the associated sums equal. For 3 ≤ k ≤ n-2 ,consider SH(n k+1) - SH(n k) = (2n+3) - 2k ≥(2n+3)-2(n-2)=7. As SH(n, k) is increasing with k hence if 3≤ k < t ≤ n-1 then SH(n t) - SH(n k) must be at least 7. We will prove that SH(n 2n-t) for any 3≤ t ≤ n-1 there is no k belongs to {3, 4, …. , n-1} such that SH(n 2n-t) = SH(n k). Now SH(n 2n-t) = SH(n t) -3 is readily available. As SH(n, k) is increasing with k hence suppose for some k belongs to{3, 4, …. , t-1} such that SH(n 2n-t) = SH(n k). Therefore SH(n k) = SH(n t) -3 therefore SH(n t) - SH(n k) = 3 which is a contradiction. Therefore no two vertices at the height n have same associated sums.

Thus the labeling is anti-magic labeling ■

One example of the above anti-magic labeling is shown below.



TT(6)

Theorem 3: For all odd natural numbers n>3, TT(n) is anti-magic.

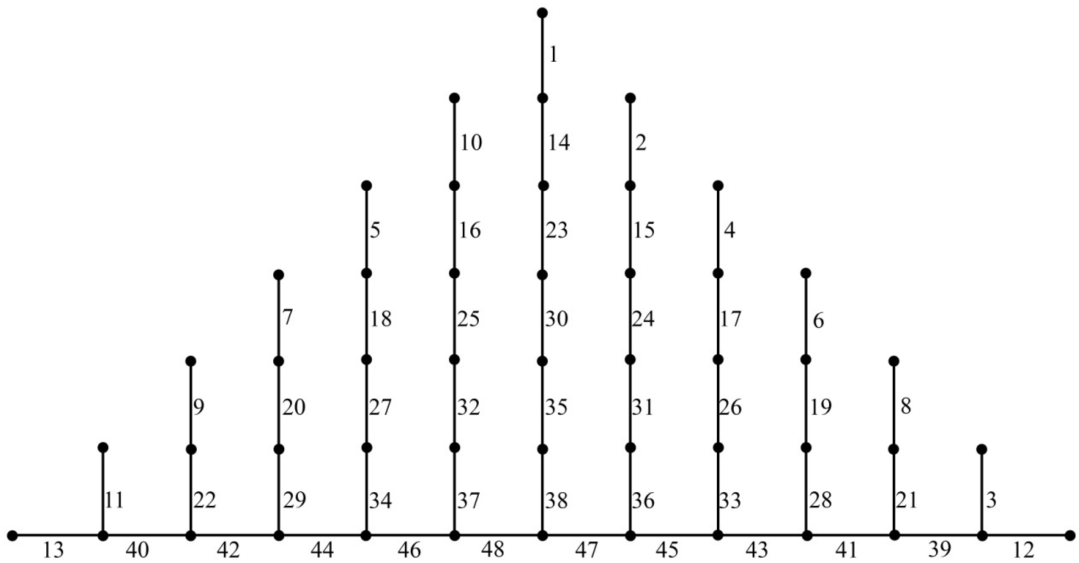
Poof: Label the edges as follows.

H(11)H(2 2) = 1

For j=1 to n-1,

H(j 1)H(j+1 2) = 2j-1, H(j 2j-1)H(j+1 2j) = 2(j-1) [Note these are the labels of pendant edges]

For j=2 to n-2 and for i=1 to (n-1)-j,
H(j j)H(j+1 j+1) = $2n(j-1)-j^2+4$
H(j+i j)H(j+i+1 j+1) = $2n(j-1)-j^2+4+2i$
H(j+i j+2i)H(j+i+1 j+2i+1) = $2n(j-1)-j^2+4+2i-1$
H(n-1 n-1)H(n n) = 2n(n-3)-(n-2)2+7
H(n 1)H(n 2) = 2(n-1)+1
H(n 2n-2)H(n 2n-1) = 2(n-1)
For k=2 to (n-1)
H(n k)H(n k+1) = 2n(n-2)-(n-1)2+4+2(k-1)
H(n 2n-(k+1))H(n 2n-k) = 2n(n-2)-(n-1)2+4+2(k-1)-1
Swap the labels of H(2 1)H(3 2) and H(n-1 2n-3)H(n 2n-2).
By proceeding similarly as above it can be seen that the above labeling is anti-magic.■
One example of the above anti-magic labeling is shown below.



TT(7)

Theorem 4: TT(n) is anti-magic for all natural numbers n>1.
Proof: It follows by combining the theorems 1, 2, 3■

References:

1. S.D. Deo, Ganesh Vishwas Joshi(2015),Conjecture for Lower Bounds of Achromatic Indices of Specially Constructed Trees. International Journal of Multidisciplinary Approach and Studies Volume 2, Issue 5,27-32
2. Gallian, J. (2019). A dynamic survey of graph labeling. Electron. J. Comb. DS6: 1–533.

3. Hartsfield, N, Ringel, G. (1990). Pearls in Graph Theory. Boston, MA: Academic Press.
4. G Joshi and S Nemani 2021 Anti-magic labeling of card House graph, J. Phys.: Conf. Ser. 1913 012118
5. Miller, M., Phanalasy, O., Ryan, J., &Rylands, L. J. (2014). A note on antimagic labelings of trees. Bulletin Of The Institute Of Combinatorics And Its Applications, 72, 94-100.
6. G Sethuraman, K.M. Shermily. Antimagic labeling of new classes of trees. AKCE International Journal of Graphs and Combinatorics Vol18,2021 Issue 2
7. Shang, J. L. (2015). Spiders are antimagic. Ars Comb. 118: 367–372.