A Survey of the Mennicke Symbol on $Um_2(R, I)$

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Abstract

In this paper, we study of Mennicke symbol on $Um_2(R, I)$ with values in classical group, where $Um_2(R, I)$will denote the set of all unimodular rows w. r. t. I.

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1. Introduction

Let R be a commutative ring with unity, and I an ideal of R. The kernel of the induced determinant map from $K_1(R, I) \to R^{\times}$is denoted as $SK_1(R, I)$, where $K_1(R)$ is defined as $GL(R)/E(R)$. It follows that the mapping $\phi: SL_2(R) \to SK_1(R)$ is surjective.

For an element $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, it is observed from [2] that $\phi(\alpha)$ depends solely on the first row (a, b). Therefore, $\phi(\alpha)$ can be expressed as $\phi(\alpha) = \begin{pmatrix} b \\ a \end{pmatrix}$. Define W as the set $\{(a, b) \mid aR + bR = R\}$, representing the first rows of such matrices α. Then, W coincides with $Um_2(R)$. According to Mennicke, the mapping [ ]:$Um_2(R) \to SK_1(R)$ demonstrates bimultiplicativebehaviour in (a, b). Such mappings are referred to as "Mennicke symbols". This paper undertakes an examination of the Mennicke symbol on $Um_2(R, I)$ with values in the relative special linear group.

2. Preliminaries

Let R be a commutative ring with 1 and I be an ideal of R.The group $GL_r(R)$consisting of $r \times r$ invertible matrices with entries in R is called the**general linear group**. The subgroup of $GL_r(R)$ which consists of invertible matrices of determinant 1 is called the **special linear group**and is denoted by $SL_r(R)$.

Let $E_r(R)$ denote the subgroup of $SL_r(R)$ consisting of all elementary matrices, i.e. those matrices which are a finite product of the elementary generators $E_{ij}(\lambda) = I_r + e_{ij}(\lambda), 1 \le i \ne j \le r, \lambda \in R,$ where $e_{ij}(\lambda) \in M_r(R)$ has at most one non-zero entry λ in its (i, j)-th position. In [5], it is proved that $E_r(R)$ is a normal subgroup of $GL_r(R)$, for $r \ge 3$.

One has the natural embedding $\alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$of $GL_r(R)$ into $GL_{r+1}(R)$, for any r. The union ofthe resulting sequence

$$GL_1(R) \subset GL_2(R) \subset \cdots \subset GL_r(R) \subset GL_{r+1}(R) \subset \cdots$$

is called the **infinite general linear group**$GL(R)$, i.e $GL(R) = \bigcup_{r \ge 1} GL_r(R)$.

Similarly, we have the subgroups $E(R) = \bigcup_{r\geq 1} E_r(R)$, $SL(R) = \bigcup_{r\geq 1} SL_r(R)$ of $GL(R)$. By ([4], Lemma 2.3), we have $E(R) = [E(R), E(R)] = [GL(R), GL(R)]$. In particular $E(R)$ is a normal subgroup of $GL(R)$ and the group $K_1(R) = \frac{GL(R)}{E(R)}$ is called the **Whitehead group** of the ring R.

Further we note that the determinant map $GL_r(R) \to R^*$ induces a map det : $K_1(R) \to R^*$ given by $\alpha E(R) \mapsto det\alpha$. The kernel of this map is $SL(R)/E(R)$ and is denoted by $SK_1(R)$. That is,

$$SK_1(R) = \frac{SL(R)}{E(R)}.$$

The group $GL_r(R, I)$ consisting of all $r \times r$ invertible matrices, which is congruent to identity modulo I, is called the relative **general linear group**, i.e.

$$GL_r(R, I) = \{A \in GL_r(R) | A \equiv I_r \text{mod} I\}.$$

The subgroup of $GL_r(R, I)$ consisting of $r \times r$ invertible matrices of determinant 1, which is congruent to identity modulo I is called the relative **special linear group** and is denoted by $SL_r(R, I)$, i.e.

$$SL_r(R, I) = \{A \in SL_r(R) | A \equiv I_r \text{ mod} I\}.$$

Let $E_r(R, I)$ denote the smallest normal subgroup of $E_r(R)$ containing the element $E_{21}(x)$, with $x \in I$. In ([7], §2), it is proved that $E_r(R, I)$ is generated by the elements $E_{ij}(a)E_{ji}(x)E_{ij}(-a)$, with $a \in R, x \in I, 1 \leq i \neq j \leq r$, when $r \geq 3$.

In relative case we have, $E(R, I) = \bigcup_{r\geq 1} E_r(R, I)$, $SL(R, I) = \bigcup_{r\geq 1} SL_r(R, I)$, $GL(R, I) = \bigcup_{r\geq 1} G_r(R, I)$ and one has $E(R, I) = [E(R), E(R, I)] = [GL(R), GL(R, I)]$. In particular $E(R, I)$ is normal subgroup of $GL(R, I)$ and we have the relative Whitehead groups

$$K_1(R, I) = \frac{GL(R, I)}{E(R, I)} \text{ and } SK_1(R, I) = \frac{SL(R, I)}{E(R, I)}$$

A row $v = (v_1, v_2, \dots, v_r) \in R^r$ is said to be **unimodular** (of length r) if there exists elements $w_1, w_2, \dots, w_r$ in R such that $\langle v, w \rangle = v \cdot w^T = v_1w_1 + v_2w_2 + \cdots + v_rw_r = 1$. $Um_r(R)$ will denote the set of all unimodular rows $v \in R^r$, i.e.

$$Um_r(R) = \{v \in R^r | \langle v, w \rangle = 1, \text{ for some } w \in R^r\}.$$

A row $v = (v_1, v_2, \dots, v_r) \in R^r$ is said to be **relativeunimodular** w. r. t. I if $v_1 = 1 \text{ mod } I, v_i \in I$, for $i > 1$, and there exists elements $w_1, w_2, \dots, w_r$ in R such that $\langle v, w \rangle = v_1w_1 + v_2w_2 + \cdots + v_rw_r = 1$. $Um_r(R, I)$ will denote the set of all relative unimodular rows w. r. t. I, i.e.

$$Um_r(R, I) = \{v \in R^r | v \equiv (1, 0, \dots, 0) \text{ mod } I, \langle v, w \rangle = 1, \text{ for some } w \in R^r\}.$$

Lemma 1:([4], Lemma 4.3) Let R be retract of B and $\pi : B \twoheadrightarrow R$. If $J = ker(\pi)$. Then, $E_r(B, J) = E_r(B) \cap SL_r(B, J)$.

The Excision ring ($\mathbb{Z} \oplus \boldsymbol{I}$): If I is an ideal in R, then one can construct the ring $\mathbb{Z} \oplus I$ with addition, component wise addition, i.e. $(n, i) + (m, j) = (n + m, i + j)$ and the multiplication$(n, i)(m, j) = (mn, nj + mi + ij)$, for $m, n \in \mathbb{Z}$, $i, j \in I$. The maximal spectrum of the ring $\mathbb{Z} \oplus I$ is noetherian, being the union of finitely many subspaces of dimension $\leq dim(R)$.There is a natural homomorphism $\varphi : \mathbb{Z} \oplus I \to R$ given by $(m, i) \mapsto m + i \in R$.If I is an ideal of ring R, one can similarly define the Excision ring $R \oplus I$.

Notation: We use the following notations:

(1) Let $A = (a_{ij})_{3\times 3} \in SL_3(R, I)$, where $a_{ij} \in I$ if $i \neq j$ and $a_{ii} \equiv 1 \text{mod } I$. Then $a_{ii} = 1 + b_{ii}$for some$b_{ii} \in I$, for $1 \leq i \leq 3$. We denote $\tilde{A} = (\widetilde{a_{ij}})$, where

$$\widetilde{a_{ij}} = \begin{cases} (0, a_{ij}) & if\ i \neq j \\ (1, b_{ii}) & if\ i = j \end{cases}$$

and thus,

$$\tilde{A} = \begin{pmatrix} (1, b_{11}) & (0, a_{12}) & (0, a_{13}) \\ (0, a_{21}) & (1, b_{22}) & (0, a_{23}) \\ (0, a_{31}) & (0, a_{32}) & (1, b_{33}) \end{pmatrix} \in SL_3(R \oplus I).$$

Also $\tilde{A} \equiv \widetilde{I_n} \text{mod } (0 \oplus I)$ and therefore $\tilde{A} \in SL_3(R \oplus I, 0 \oplus I)$.

Lemma2: Let I be an ideal of ring$R$ and $\psi : R \oplus I \to R$be the natural ring homomorphism given by $\psi(x, i) = x + i$. Then $\phi: E_3(R \oplus I, 0 \oplus I) \to E_3(R, I)$,defined by $\phi(\tilde{\alpha}) = \left(\psi(\widetilde{\alpha_{ij}})\right)$, where $\tilde{\alpha} = (\widetilde{\alpha_{ij}})$, is a group homomorphism induced by ψ and $\phi(\tilde{\alpha}) = \alpha$.

Proof: Let $\tilde{A}, \tilde{B} \in E_3(R \oplus I, 0 \oplus I)$, say$\tilde{A} = (\widetilde{a_{ij}})$ and $\tilde{B} = (\widetilde{b_{ij}})$. Then there exist $c_{ii}, d_{ii} \in I$, $1 \leq i \leq 3$ such that $\widetilde{a_{ii}} = (1, c_{ii})$, $\widetilde{b_{ii}} = (1, d_{ii})$ and for $i \neq j$, $\widetilde{a_{ij}} = (0, a_{ij})$, $\widetilde{b_{ij}} = (0, b_{ij})$.Then

$$\tilde{A} = \begin{pmatrix} (1, c_{11}) & (0, a_{12}) & (0, a_{13}) \\ (0, a_{21}) & (1, c_{22}) & (0, a_{23}) \\ (0, a_{31}) & (0, a_{32}) & (1, c_{33}) \end{pmatrix}, \tilde{B} = \begin{pmatrix} (1, d_{11}) & (0, b_{12}) & (0, b_{13}) \\ (0, b_{21}) & (1, d_{22}) & (0, b_{23}) \\ (0, b_{31}) & (0, b_{32}) & (1, d_{33}) \end{pmatrix}$$

and $\tilde{A}\tilde{B} = (\widetilde{e_{ik}})$, where $\widetilde{e_{ik}} = \sum_{j=1}^{3} \widetilde{a_{ij}}\widetilde{b_{jk}}$, for $1 \leq i, k \leq 3$. One can write $\widetilde{e_{ik}}$explicitly as

$$\widetilde{e_{ik}} = \begin{cases} (1, f_{ii}) & \text{if } i = k \\ (0, e_{ik}) & \text{if } i \neq k \end{cases}$$

where $f_{ii} = c_{ii} + d_{ii} + c_{ii}d_{ii} + \sum_{j \neq i} a_{ij}b_{jk}$and $e_{ik} = a_{ik} + b_{ik} + c_{ii}b_{ik} + d_{ii}a_{ik} + \sum_{\substack{j \neq i, \\ j \neq k}} a_{ij}b_{jk}$. Clearly,$f_{ii}$, $e_{ik} \in I$ and by the definition of ϕ,

$$\phi(\tilde{A}\tilde{B}) = \phi((\widetilde{e_{ik}})) = \begin{pmatrix} 1 + f_{11} & e_{12} & e_{13} \\ e_{21} & 1 + f_{22} & e_{23} \\ e_{31} & e_{32} & 1 + f_{33} \end{pmatrix}$$

$$= \begin{pmatrix} 1+ c_{11} & a_{12} & a_{13} \\ a_{21} & 1+c_{22} & a_{23} \\ a_{31} & a_{32} & 1+c_{33} \end{pmatrix} \begin{pmatrix} 1+ d_{11} & b_{12} & b_{13} \\ b_{21} & 1+d_{22} & b_{23} \\ b_{31} & b_{32} & 1+d_{33} \end{pmatrix}$$

$= \phi(\tilde{A})\phi(\tilde{B})$.

Also $\phi(\tilde{A}) = \begin{pmatrix} 1+ c_{11} & a_{12} & a_{13} \\ a_{21} & 1+c_{22} & a_{23} \\ a_{31} & a_{32} & 1+c_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A.$

Lemma 3: Let I be an ideal of ring R. If $\tilde{\alpha} \in E_3(R \oplus I) \cap SL_3(R \oplus I, 0 \oplus I)$ then $\alpha \in E_3(R, I)$.

Proof: Since $R$is a retract of the ring $R \oplus I$ and the projection map$\pi: R \oplus I \twoheadrightarrow R$ defined as $(x, i) \mapsto x$ with $\ker(\pi) = 0 \oplus I$, by Lemma 1, $\tilde{\alpha} \in E_3(R \oplus I, 0 \oplus I)$. Hence by Lemma 2, $\alpha \in E_3(R, I)$.

3.Mennicke Symbol

Let I be a non-zero ideal of ring$R$. We say that two pairs (a_1, b_1) and (a_2, b_2) in R^2, are I-equivalent if one is obtained from other by a finite sequence of elementary transformations of the type,

$(a, b) \longmapsto (a, b + ta)$ $(t \in I)$ and $(a, b) \longmapsto (a + bt, b)$ $(t \in R)$, denoted as

$$(a_1, b_1) \sim_I (a_2, b_2).$$

Definition 4 (Mennicke symbol) A Mennicke symbol on $Um_2(R, I)$ is the function

$$[\,] : Um_2(R, I) \longrightarrow C,$$

$$(a, b) \longmapsto \begin{bmatrix} b \\ a \end{bmatrix}$$

where C is a group which satisfies,

MS1: $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$ and $\begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}$ if $(a_1, b_1) \sim_I (a_2, b_2)$.

MS2a: If $(a, b_1), (a, b_2) \in Um_2(R, I)$ then $\begin{bmatrix} b_1 b_2 \\ a \end{bmatrix} = \begin{bmatrix} b_1 \\ a \end{bmatrix}\begin{bmatrix} b_2 \\ a \end{bmatrix}$.

MS2a: If $(a_1, b), (a_2, b) \in Um_2(R, I)$ then $\begin{bmatrix} b \\ a_1 a_2 \end{bmatrix} = \begin{bmatrix} b \\ a_1 \end{bmatrix}\begin{bmatrix} b \\ a_2 \end{bmatrix}$.

We prove the following preliminary result.

Lemma 5: Let I be an ideal of ring R. Let $A = (a_{ij}) \in SL_3(R, I)$ and $\varepsilon_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $\varepsilon_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in E_3(R)$. Then, for $1 \le i \le 3, A \equiv \varepsilon_i\, A\, \varepsilon_i^{-1} \bmod E_3(R, I)$.

Proof: Since $A = (a_{ij}) \in SL_3(R, I)$, we write $A = \begin{pmatrix} 1+ b_{11} & a_{12} & a_{13} \\ a_{21} & 1+b_{22} & a_{23} \\ a_{31} & a_{32} & 1+b_{33} \end{pmatrix}$

where $a_{ij},\ b_{ii} \in I, 1 \le i \ne j \le 3$. A direct computation shows

$$\varepsilon_1 A \varepsilon_1^{-1} = \begin{pmatrix} a_{22} & a_{23} & -a_{21} \\ a_{32} & a_{33} & -a_{31} \\ -a_{12} & -a_{13} & a_{11} \end{pmatrix} = B_1 \text{(say)}.$$

But by definition, $\tilde{A} = \begin{pmatrix} (1, \mathrm{b}_{11}) & (0, a_{12}) & (0, a_{13}) \\ (0, a_{21}) & (1, \mathrm{b}_{22}) & (0, \mathrm{a}_{23}) \\ (0, \mathrm{a}_{31}) & (0, \mathrm{a}_{32}) & (1, \mathrm{b}_{33}) \end{pmatrix} \in SL_3(R \oplus I)$ and $\widetilde{\varepsilon_1} = \begin{pmatrix} (0,0) & (-1,0) & (0,0) \\ (0,0) & (0,0) & (-1,0) \\ (1,0) & (0,0) & (0,0) \end{pmatrix}, \widetilde{\varepsilon_1}^{-1} = \begin{pmatrix} (0,0) & (0,0) & (1,0) \\ (-1,0) & (0,0) & (0,0) \\ (1,0) & (-1,0) & (0,0) \end{pmatrix} \in E_3(R \oplus I)$.

Then,

$$\widetilde{\varepsilon_1} \tilde{A} \widetilde{\varepsilon_1}^{-1} = \begin{pmatrix} (1, \mathrm{b}_{22}) & (0, a_{23}) & (0, -a_{21}) \\ (0, a_{32}) & (1, \mathrm{b}_{33}) & (0, -\mathrm{a}_{31}) \\ (0, -\mathrm{a}_{12}) & (0, -\mathrm{a}_{13}) & (1, \mathrm{b}_{11}) \end{pmatrix} = \begin{pmatrix} \widetilde{a_{22}} & \widetilde{a_{23}} & -\widetilde{a_{21}} \\ \widetilde{a_{32}} & \widetilde{a_{33}} & -\widetilde{a_{31}} \\ -\widetilde{a_{12}} & -\widetilde{a_{13}} & \widetilde{a_{11}} \end{pmatrix} = \widetilde{B_1}$$

and thus $\widetilde{\varepsilon_1} \tilde{A} \widetilde{\varepsilon_1}^{-1} \widetilde{B_1}^{-1} = \widetilde{I_n}$.

Since$E_3(R \oplus I)$ is normal subgroup of $SL_3(R \oplus I)$, $\tilde{A} \widetilde{\varepsilon_1}^{-1} = \tilde{\varepsilon} \tilde{A}$, for some$\tilde{\varepsilon} \in E_3(R \oplus I)$ and therefore $\widetilde{\varepsilon_1} \tilde{\varepsilon} \tilde{A} \widetilde{B_1}^{-1} = \widetilde{I_n}$. Hence $\tilde{A} \widetilde{B_1}^{-1} = \tilde{\varepsilon}^{-1} \widetilde{\varepsilon_1}^{-1} \in E_3(R \oplus I)$.

Also,$\tilde{A} \widetilde{B_1}^{-1} \in SL_3(R \oplus I, 0 \oplus I)$as $\tilde{A}, \widetilde{B_1} \in SL_3(R \oplus I, 0 \oplus I)$.Hence, $\tilde{A} \widetilde{B_1}^{-1} \in \mathrm{E}_3(\mathrm{R} \oplus \mathrm{I}) \cap SL_3(R \oplus I, 0 \oplus I)$.By Lemma3,$AB_1^{-1} \in E_3(R, I)$,i.e. $A \equiv B_1 \bmod E_3(R, I)$ or $A \equiv \varepsilon_1 A \varepsilon_1^{-1} \bmod E_3(R, I)$. Similarly,we can prove for $\varepsilon_2$and ε_3.

Lemma 6: Let I be an ideal of ring R. Let $A = (a_{ij}) \in SL_3(R, I)$and $\varepsilon_1 = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$, $\varepsilon_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} \in E_3(R)$, where$t \in R$. Then, for $4 \leq i \leq 6, A \equiv \varepsilon_i A \varepsilon_i^{-1} \bmod E_3(R, I)$.

Proof: We will follow the same argument used in Lemma 5. By a simple computation we have,

$$\varepsilon_4 A \varepsilon_4^{-1} = \begin{pmatrix} a_{11} - t a_{12} & a_{12} & a_{13} \\ a_{21} + t a_{11} - t a_{22} - t^2 a_{12} & a_{22} + t a_{12} & a_{23} + t a_{13} \\ a_{31} - t a_{32} & a_{32} & a_{33} \end{pmatrix} = B_4 \text{ (say)}.$$

Hence,

$$\widetilde{\varepsilon_4} \tilde{A} \widetilde{\varepsilon_4}^{-1} = \begin{pmatrix} \widetilde{a_{11}} - \tilde{t} \widetilde{a_{12}} & \widetilde{a_{12}} & \widetilde{a_{13}} \\ \widetilde{a_{21}} + \tilde{t} \widetilde{a_{11}} - \tilde{t} \widetilde{a_{22}} - \tilde{t}^2 \widetilde{a_{12}} & \widetilde{a_{22}} + \tilde{t} \widetilde{a_{12}} & \widetilde{a_{23}} + \tilde{t} \widetilde{a_{13}} \\ \widetilde{a_{31}} - \tilde{t} \widetilde{a_{32}} & \widetilde{a_{32}} & \widetilde{a_{33}} \end{pmatrix} = \widetilde{B_4}.$$

and therefore,$\widetilde{\varepsilon_4} \tilde{A} \widetilde{\varepsilon_4}^{-1} \widetilde{B_4}^{-1} = \widetilde{I_n}$.Since$E_3(R \oplus I)$ is normal subgroup of $SL_3(R \oplus I)$, $\widetilde{\varepsilon_4} \tilde{\varepsilon} \tilde{A} \widetilde{B_4}^{-1} = \widetilde{I_n}$, for some $\tilde{\varepsilon} \in E_3(R \oplus I)$. Hence $\tilde{A} \widetilde{B_4}^{-1} = \tilde{\varepsilon}^{-1} \widetilde{\varepsilon_4}^{-1} \in E_3(R \oplus I)$.Since,$\tilde{A}, \widetilde{B_4} \in SL_3(R \oplus I, 0 \oplus I)$,$\tilde{A} \widetilde{B_4}^{-1} \in SL_3(R \oplus I, 0 \oplus I)$andhence,$\tilde{A} \widetilde{B_4}^{-1} \in \mathrm{E}_3(\mathrm{R} \oplus \mathrm{I}) \cap SL_3(R \oplus I, 0 \oplus I)$.By Lemma 3, $AB_4^{-1} \in E_3(R, I)$,i.e. $A \equiv B_4 \bmod E_3(R, I)$ or $A \equiv \varepsilon_4 A \varepsilon_4^{-1} \bmod E_3(R, I)$. Similarly, we can prove for ε_5 and ε_6.

Lemma 7: Let I be an ideal of ring R. Let $\alpha = \delta_1 \perp 1$ and $\beta = \delta_2 \perp 1$, where$\delta_i = \begin{pmatrix} a & b_i \\ c_i & d_i \end{pmatrix} \in SL_2(R, I)$, for $i = 1, 2$. Then $\alpha\beta \equiv \gamma \bmod E_3(R, I)$, where $\gamma = \begin{pmatrix} a & b_1 b_2 & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Proof: For $\varepsilon_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 1 & 0 & c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \varepsilon_3 = \begin{pmatrix} 1 & 0 & 0 \\ -c_1 d_2 & 1 & 0 \\ b_2 & 0 & 1 \end{pmatrix}$, $\varepsilon_4 = \begin{pmatrix} 1 & -b_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\varepsilon_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, we have

$\varepsilon_5(\varepsilon_3\varepsilon_2\alpha\,(\varepsilon_1\,\beta\,\varepsilon_1^{-1})\varepsilon_4)\varepsilon_5^{-1}$

$$= \varepsilon_5\varepsilon_3\varepsilon_2\alpha \begin{pmatrix} d_2 & 0 & -c_2 \\ 0 & 1 & 0 \\ -b_2 & 0 & a \end{pmatrix} \varepsilon_4\varepsilon_5^{-1} = \varepsilon_5\varepsilon_3\varepsilon_2 \begin{pmatrix} ad_2 & b_1 & -ac_2 \\ c_1 d_2 & d_1 & -c_1 c_2 \\ -b_2 & 0 & a \end{pmatrix} \varepsilon_4\varepsilon_5^{-1}$$

$$= \varepsilon_5 \begin{pmatrix} 1 & b_1 & 0 \\ 0 & d_1 - b_1 c_1 d_2 & -c_1 c_2 \\ 0 & b_1 b_2 & a \end{pmatrix} \varepsilon_4\varepsilon_5^{-1} = \varepsilon_5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & d_1 - b_1 c_1 d_2 & -c_1 c_2 \\ 0 & b_1 b_2 & a \end{pmatrix} \varepsilon_5^{-1}$$

$$= \begin{pmatrix} a & b_1 b_2 & 0 \\ -c_1 c_2 & d_1 - b_1 c_1 d_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \gamma.$$

By Lemma 5, $\varepsilon_3\varepsilon_2\alpha\,(\varepsilon_1\,\beta\,\varepsilon_1^{-1})\varepsilon_4 \equiv \gamma \bmod E_3(R, I)$.Since, $\varepsilon_2, \varepsilon_3, \varepsilon_4 \in E_3(R, I)$, threrfore $\alpha\,(\varepsilon_1\,\beta\,\varepsilon_1^{-1}) \equiv \gamma \bmod E_3(R, I)$.Butby Lemma 5, $\beta \equiv \varepsilon_1\,\beta\,\varepsilon_1^{-1} \bmod E_3(R, I)$and hence $\alpha\beta \equiv \gamma \bmod E_3(R, I)$.

Lemma 8: Let I be an ideal of ring R. Let $\alpha = \delta_1 \perp 1$ and$\beta = \delta_2 \perp 1$, where $\delta_i = \begin{pmatrix} a_i & b \\ c_i & d_i \end{pmatrix} \in SL_2(R, I)$, for $i = 1, 2$. Then $\alpha\beta \equiv \gamma \bmod E_3(R, I)$,where $\gamma = \begin{pmatrix} a_1 a_2 & b & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Proof: For $\varepsilon_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -a_1 \\ 0 & 0 & 1 \end{pmatrix}, \varepsilon_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a_1 - 1 \\ 0 & 0 & 1 \end{pmatrix}, \varepsilon_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \varepsilon_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & d_2 - 1 \\ 0 & 0 & 1 \end{pmatrix}, \varepsilon_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c_1 a_2 & 1 - d_1 & 1 \end{pmatrix}$, we have

$$\varepsilon_5\varepsilon_4\varepsilon_3\varepsilon_2\alpha(\varepsilon_1\,\beta\,\varepsilon_1^{-1})\varepsilon_2^{-1}\varepsilon_4^{-1}\varepsilon_6 = \varepsilon_5\varepsilon_4\varepsilon_3\varepsilon_2\alpha \begin{pmatrix} a_2 & 0 & -b \\ 0 & 1 & 0 \\ -c_2 & 0 & d_2 \end{pmatrix} \varepsilon_2^{-1}\varepsilon_4^{-1}\varepsilon_6$$

$$= \varepsilon_5\varepsilon_4\varepsilon_3\varepsilon_2 \begin{pmatrix} a_1 a_2 & b & -a_1 b \\ c_1 a_2 & d_1 & -c_1 b \\ -c_2 & 0 & d_2 \end{pmatrix} \varepsilon_2^{-1}\varepsilon_4^{-1}\varepsilon_6$$

$$= \varepsilon_5\varepsilon_4\varepsilon_3 \begin{pmatrix} a_1a_2 & b & 0 \\ c_1a_2 + a_1c_2 & d_1 & 1 - a_1d_2 \\ -c_2 & 0 & d_2 \end{pmatrix} \varepsilon_4^{-1}\varepsilon_6$$

$$= \varepsilon_5\varepsilon_4 \begin{pmatrix} a_1a_2 & b & 0 \\ c_1a_2 + c_2 & d_1 & 1 - d_2 \\ -c_2 & 0 & d_2 \end{pmatrix} \varepsilon_4^{-1}\varepsilon_6$$

$$= \varepsilon_5 \begin{pmatrix} a_1a_2 & b & 0 \\ c_1a_2 + c_2 & d_1 + d_2 - 1 & 1 - d_2 \\ c_1a_2 & d_1 - 1 & 1 \end{pmatrix} \varepsilon_6$$

$$= \begin{pmatrix} a_1a_2 & b & 0 \\ c_1a_2d_2 + c_2 & d_1d_2 & 0 \\ c_1a_2 & d_1 - 1 & 1 \end{pmatrix} \varepsilon_6$$

$$= \begin{pmatrix} a_1a_2 & b & 0 \\ c_1a_2d_2 + c_2 & d_1d_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that, $\varepsilon_3, \varepsilon_5, \varepsilon_6 \in E_3(R, I)$and by Lemma 5and Lemma 6, $\alpha\,(\varepsilon_1\,\beta\,\varepsilon_1^{-1}) \equiv \gamma$ mod $E_3(R, I)$.Again by Lemma 5, $\beta \equiv \varepsilon_1\,\beta\,\varepsilon_1^{-1}$ mod $E_3(R, I)$ and hence $\alpha\beta \equiv \gamma$ mod $E_3(R, I)$.

Theorem 9: Let $(a, b) \in Um_2(R, I)$ and$c \in I, d \in R$and $d \equiv 1$ mod I be such that $ad - bc = 1$. Define$ms : Um_2(R, I) \to SL_3(R, I)/E_3(R, I)$ by

$$ms\begin{pmatrix} b \\ a \end{pmatrix} = \text{class of} \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ in } \frac{SL_3(R,\ I)}{E_3(R,\ I)}$$

Then ms is a Mennicke symbol.

Proof: MS1: Let $(a, b) \in Um_2(R, I)$ and$c \in I, d \in R$and $d \equiv 1$ mod Ibe such that $ad - bc = 1$. Note that by definition, $ms\begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$ mod $E_3(R, I)$.

Since,$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & at + b & 0 \\ c & ct + d & 0 \\ 0 & 0 & 1 \end{pmatrix}$, for all $t \in I$, we have,

$$ms\begin{pmatrix} b \\ a \end{pmatrix} = ms\begin{pmatrix} at + b \\ a \end{pmatrix}, \text{ for all } t \in I.$$

Since,

$$\begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a + tb & b & 0 \\ -at + c - bt^2 + dt & -bt + d & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for all $t \in R$, by Lemma 6,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} a + tb & b & 0 \\ -at + c - bt^2 + dt & -bt + d & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ mod } E_3(R, I), \text{ for all } t \in R$$

and hence,

$$ms\binom{b}{a} = ms\binom{b}{a+bt}, \text{ for all } t \in R.$$

MS2a: Let $(a, b_1), (a, b_2) \in Um_2(R, I)$ and $c_1, c_2 \in I$, $d_1, d_2 \in R$, $d_1 \equiv 1$ mod $I$and $d_2 \equiv 1$ mod Ibe such that $ad_1 - c_1b_1 = 1$ and$ad_2 - c_2b_2 = 1$. By definition

$$ms\binom{b_1}{a} = \begin{pmatrix} a & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ mod } E_3(R,I) \text{ and } ms\binom{b_2}{a} = \begin{pmatrix} a & b_2 & 0 \\ c_2 & d_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ mod } E_3(R,I).$$

By Lemma 7,$\begin{pmatrix} a & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b_2 & 0 \\ c_2 & d_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} a & b_1b_2 & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$ mod $E_3(R,I)$ and hence,

$$ms\binom{b_1}{a} ms\binom{b_2}{a} = ms\binom{b_1b_2}{a}.$$

MS2b: Let $(a_1, b), (a_2, b) \in Um_2(R, I)$ and $c_1, c_2 \in I$, $d_1, d_2 \in R$, $d_1 \equiv 1$ mod $I$and $d_2 \equiv 1$ mod I be such that $a_1d_1 - c_1b = 1$and $a_2d_2 - c_2b = 1$. By definition

$$ms\binom{b}{a_1} = \begin{pmatrix} a_1 & b & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{mod} E_3(R,I) \text{ and } ms\binom{b}{a_2} = \begin{pmatrix} a_2 & b & 0 \\ c_2 & d_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ mod } E_3(R,I).$$

By Lemma 8, $\begin{pmatrix} a_1 & b & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_2 & b & 0 \\ c_2 & d_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} a_1a_2 & b & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$ mod $E_3(R,I)$ and hence,

$$ms\binom{b}{a_1} ms\binom{b}{a_1} = ms\binom{b}{a_1a_2}.$$

Thusms is a Mennicke symbol.

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