Prescriptive Remarks on Theorems in d-Metric and dq-Metric Spaces

Swapnil Ramesh Gadhawe
Shriram Nagar, Tumsar, MS, India

Abstract

In this research paper, we focused on theorems in d-metric and dq-metric spaces. These concepts are recently becoming remarkable research topics in the field of metric fixed point theory.

KEYWORDS: d-metric space, dq-metric space, contraction.

I. INTRODUCTION

In 2000, Hitzler [1] introduced the concept of d-metric space. In such a space the distance between similar points need not to be zero necessarily. Also Hitzler [1] proved the famous Banach contraction theorem in such a space. Dislocated metric space play a vital role in Topology, Logical Programming and Electronics Engineering etc. Zeyada, Hassan and Ahmad [2] introduced the concept of complete dq-metric space. With the passage of time many papers have been published containing fixed point results for a single and a pair of self-mappings with different type of contractive conditions in dislocated quasimetric space (see [3], [4], [5], [6], [7], [8]). Recently, Kumari et al. [11] proved a common fixed point theorem for two pairs of weakly compatible mappings in dislocated metric space.

The function d is called the metric on X. It is also sometimes called a distance function or simply a distance. Often d is omitted and one just writes X for a metric space if it is clear from the context what metric is being used. If we use the discrete metric, which induces the discrete topology, then this product topology is also discrete and d is indeed continuous (as is any map on a discrete space, so it's not very informative).

In this paper, we have given some remarks about the mistakes made by various authors during the proof of fixed point results in the setting of d-metric and dq-metric spaces. Also we have provided those references in which these type of mistakes are removed by the authors.

II. PRELIMINARIES

In this research paper R+ stand for the set of non negative real numbers.

Definition 2.1. [1]. Consider X_0 be a set which is non-empty and $d_0 : X_0 \text{ x } X_0 \to$ R+ be a function satisfying the conditions

d_1) $d_0(\chi_1, \chi_1) = 0$;

d_2) $d_0(\chi_1, \gamma_1) = d_0(\gamma_1, \chi_1) = 0$ implies that $\chi_1 = \gamma_1$

d_3) d_0 (χ_1, γ_1) = d_0 (γ_1, χ_1);

d_4) d_0 (χ_1, γ_1) ≤ d_0 (χ_1, z_1) + d_0 (z_1, γ_1) for all χ_1, γ_1 z_1 ∈ χ_0

If d_0 satisfy the conditions d_1 to d_4 then it is called metric on X_0, if d_0 satisfy conditions d_2 to d_4 then it is called *d*-metric on X_0 and if d_0 satisfy conditions d_2 and d_4 only then it is called *dq*-metric on X_0.

Clearly every metric space is a *d*-metric space but the reverse inclusion is not true as clear from the following example.

Example 2.2. Let $X_0 = R^+$ define a mapping $d_0 : X_0 \times X_0 \rightarrow R^+$ as

$$d_0 (\chi_1, \gamma_1) = \chi_1 + \gamma_1.$$

Clearly the define function is *d*-metric space but not a metric space.

Also every metric space is *dq*-metric space but the reverse is not true and every *d*-metric space is *dq*-metric space but also the reverse inclusion is not true see the following example.

Example 2.3. Let X_0 = R we define the function $d_0 : X_0 \times X_0 \rightarrow R^+$ by

$$d_0 (\chi_1, \gamma_1) = |\chi_1| \text{ for all } \chi_1, \gamma_1 \in X_0.$$

Evidently the defined function is *dq*-metric space but not a metric space nor *d*-metric space.

The following definitions are required in the sequel which can be found in ([1], [2]).

Definition 2.4. A sequence $\{\chi_n\}$ in *d*-metric (*dq*-metric) space (X_0, d_0) is called Cauchy sequence if for ∈ > 0 there exists a positive integer $n_0 \in N$ such that for $m, n > n_0$, so d_0 (χ_m, χ_n) < ∈.

Definition 2.5. A sequence $\{\chi_n\}$ is called *d*-convergent (*dq*-convergent) in (X_0, d_0) if for $n \in N$ we have

$$\lim_{n\rightarrow\infty} d_0(d_0, x) = \lim_{n\rightarrow\infty} d_0(x, x_n) = 0$$

In this case x is called the *dq*-limit of the sequence $\{x_n\}$.

Definition 2.6. A *d*-metric (*dq*-metric) space (X_0, d_0) is said to be complete if every Cauchy sequence in X converge to a point of X.

Definition 2.7. Assume that (X_0, d_0) be a *d*-metric (*dq*-metric) space. A mapping $T_0 : X_0 \rightarrow X_0$ is called contraction if there exists $0 \le \alpha < 1$ such that

$$d_0 (T_0x, T_0y) \le \alpha \cdot d_0 (x, y) \text{ for all } x, y \in X_0.$$

The following well-known results can be seen in [1] and [2].

Lemma 2.8. Limit of a convergent sequence in d-metric (dq-metric) space is unique.

Theorem 2.9. Assume that (X_0, d_0) be a d-metric (dq-metric)

space which is complete $T_0 : X_0 \to X_0$ be a contraction then T_0 has a fixed point which is unique.

III. REMARKS

The self distance between points in d-metric and dq-metric spaces need not to be zero essentially as described in the above examples. But still in some examples (such is in Example 2.3) there exists points for which self distance remain zero. So we define a set in d and dq-metric spaces as follows:

$$A = \{x \in X \; d(x, x) = 0\}$$

i.e that the self distance remain zero for those points belonging to set A. Also there exist a result through which we can brings the points of X into A.

Lemma 3.1. If $\{x_n\}$ is a d-convergent (dq-convergent) in (X, d) with d-limit (dq-limit) in $x \in X$ then $x \in A$.

The proof of the above result is quite simple.

Now we discuss some remarks regarding fixed point results in the frame work of d-metric and dq-metric spaces.

1. Shrivastava et al. [5] established the following result for a continuous self-mapping in dq-metric space.

Theorem 3.2. Assume that (X_0, d_0) be a dq-metric space and $f : X_0 \to X_0$ be a mapping which is continuous satisfying

$$d_0(fx, fy) \leq \lambda \; \frac{d_0\,(y,fy)[1+ d_0(x,fx)]}{1+ d_0\,(x,y)} \; p \,.\, d_0\,(x, y) +$$

$$\delta \; \frac{d_0\,(y,fy) + d_0(y,fx)}{1+\; d_0\,(y,fy)d_0(y,fx)}$$

for all $x, y \in X_0$ and $\lambda, \rho, \delta \geq 0$ with $\lambda + \rho + \delta < 1$. Then f has a unique fixed point.

In the proof of the above theorem after constructing a sequence $\{x_n\}$, to show that $\{x_n\}$ is a sequence which is Cauchy the term $d(x_n, x_n)$ appear in the overline contraction condition which is omitted by the authors i.e put it equal to zero. But as mention above the self distance between points in dislocated quasi-metric space need not to be zero necessarily due to which the restriction on δ also become wrong.

This contraction condition along with constant associated with it is correctly used by Sarwar et al. [4]. Also we noticed that in the statement of the theorem there is nothing about the completeness of the space but in the proof of the theorem authors used the condition of completeness.

Similar type of situation also occur in the following remark.

2. Patel et al. [7] established the following theorem in *dq*-metric space.

Theorem 3.3. Assume that (X_0, d_0) be a *dq*-metric space which is complete and suppose there exists $a_1, a_2, a_3, a_4, a_5 \geq 0$ with $a_1 + a_2 + a_3 + (a_4 + a_5) < 1$. Let $f : X_0 \llcorner X_0$ be a mapping which is continuous satisfying

$$d_0(fx_1, fy_1) \leq a_1 \cdot d_0(x_1, y_1) + a_2 \cdot d_0(x_1, fx_1) + a_3 \cdot d_0(y_1, fy_1) +$$

$$a_4 \cdot [d_0(x_1, fx_1) + d_0(y_1, fy_1)] + a_5 \cdot [d_0(x_1, fy_1) + d_0(y_1, fx_1)]$$

for all $x_1, y_1 \in X_0$. Then f has a fixed point which is unique.

Like above situation when the term $d(x_n, x_n)$ occur in the above overline contraction condition the authors put it equal to zero due to which the restriction on the constant associated with it become $2a_5$.

This mistake was properly solved by Panthi et al. [6] and Sarwar et al. [4]. Due to which the restriction on the constant become $4a_5$ which is quite true thing.

Similar mistake was done by Patel et al. [8].

3. In [3] Isufati established the following theorem in *d*-metric space.

Theorem 3.4. Assume that (X_0, d_0) be a complete *d*-metric space and $f, g : X_0 \rightarrow X_0$ be a mappings which is continuous satisfying

$$d_0(fx_1, gy_1) \leq \delta \max \{d_0(x_1, y_1), d_0(x_1, fx_1), d_0(y_1, gy_1),$$

$$d_0(x_1, gy_1) + d_0(y_1, fx_1) / 2\}$$

for all $x_1, y_1 \in X_0$ and $0 \leq \delta < 1$. Then f and g have common fixed point which is unique.

Rao and Swamy [9] provided the following example and claim that the the above theorem doesn't hold in the view of the above example.

Example 3.5. Let $X_0 = \{1, 2, 3\}$ and

$$d_0(x_1, y_1) = \begin{cases} 2 \text{ if } x_1 + y_1 \text{ is even} \\ 1 \text{ if } x_1 + y_1 \text{ is odd} \end{cases}$$

Define $f, g : X_0 \rightarrow X_0$ by

$$f1 = f2 = f3 = 1$$

and

$$gg1 = g2 = g3 = 2$$

Clearly all the axioms of the above theorem are satisfied for $\delta = \frac{1}{2} < 1$ but f and g have no fixed point which is common to both of them.

Our reservation is that: that Isufati [3] proved the theorem for continuous self-mappings while the above defined functions are obviously discontinuous.

4. Recently in 2014, Balaji et al. [10] established a common fixed point result for four self-mappings which are weakly compatible in d-metric space and deduce some corollaries from it.

But it is quite evident that in 2012, Kumari et al. proved exactly this theorem and deduced similar corollaries from it in their paper [11].

IV. CONCLUSION

We observe that the literature review for our research work are not too healthy because in many cases the author remain unable to find whether the problem which he is solving is already done or not. We also notice that during the research work we doesn't study things deeply due to which authors made mistakes which are quite obvious.

REFERENCES

[1] P. Hitzler, *Generalized metrics and Topology in Logic Programming Semantics*, Ph.D Thesis, National Univeristy of Ireland, University College, Cork, 2001.

[2] F. M. Zeyada, G. F. Hassan and M. A. Ahmad, "A generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi-metric space", *Arabian Journal of Science and Engineering,* vol. 31, pp. 111–114, 2005.

[3] A. Isufati, "Fixed point theorem in dislocated quasi-metric spaces", *Applied Mathematical Sciences*, vol. 4, pp. 217–223, 2010.

[4] M. Sarwar, M. U. Rahman and G. Ali, "Some fixed point results in dislocated quasi-metric (dq-metric) spaces", *Journal of Inequalities and Applications*, vol. 278, pp. 1–11, 2014.

[5] M. Shrivastava, Q. K. Qureshi and A. D. Singh, "A fixed point theorem for continuous mapping in dislocated quasi-metric space", *International Journal of Theoratical and Applied Sciences*, vol. 4, pp. 39–40, 2012.

[6] D. Panthi, K. Jha and G. Porru, A fixed point theorem in dislocated quasimetric space, *American Journal of Mathematics and Statistic*, **3**(2013), 153 - 156.

[7] S. T. Patel and M. Patel, Some results of fixed point theorem in dislocated quasi-metric space, *Journal of Research in Modern Engineering and Emerging Technology*, **1**(2013), 20 - 24.

[8] S. T. Patel, V. C. Makvana and C. H. Patel, "Some results of fixed point theorem in dislocated quasi-metric space of integral type", *Scholars Journal of Engineering and Technology*, vol. 2, pp. 91–96, 2014.

[9] K. P. R. Rao and P. R. Swamy, “A coincidence point theorem for four mappings in dislocated metric space”, *International Journal of Contemporary Mathematical Sciences*, vol. 6, pp. 1675–1680, 2011.

[10] R. W. Balaji, R. Bhardwaj and B. K. Singh, “A common fixed point theorem in dislocated metric space”, *International Journal of Engineering Research and Development*, vol. 10, pp. 14–17, 2014.

[11] P. S. Kumari, V. V. Kumar and I. R. Sarma, “New version for Hardy-Rogers type mapping in dislocated metric space”, *International Journal of Basic and Applied Sciences*, vol. 1, pp. 609–617, 2012.