M/M/1 Model with K-Phase of Service under N-Policy

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Abstract

The model developed has been designed to analyze the behavior of M/M/1 queueing system with k phases of services wherein 1st phase service is essential and the rest are optional services. We analyze a single server Markovian queue with breakdown under N-policy which implies that server will not start service until the queue size becomes N (≥ 1). The arrival of customers in the system is in the Poisson fashion. We consider the case in which most of the customers require first "essential" service whereas a few of them may demand the second service which is "optional". The service time of essential service and optional service phases are exponentially distributed.

The noble feature of the present study is the incorporation of essential and optional (k-1) phases of services. We construct the Chapman-Kolmogorov equations by considering the birth death process. We present a matrix geometric approach for the solution purpose and develop an algorithm for computing the steady state probability vector which is further used to compute various performance measures. Numerical illustration is provided to examine the tractability of the algorithmic procedure.

KEYWORDS: N-policy, Essential service, Optional phase service, Phase repair, Server breakdown, Matrix geometric method.

1. Introduction

In modern cellular mobile communication network, the quality of service (QoS) experienced subjectively by individual customers is the crucial factor to determine the system performance. Thus, the incorporation of customer's behavior is essential in order to gain realistic input for network planning. With the advances in science and technology, telephony services over IP networks are the promising trends in telecommunication business. The aim of the present service system is to provide continuous and better service. In queueing systems, such as computer, communication, manufacturing, production systems, etc., the situation of server breakdown affects the overall production capacity and quality of the systems.

The modeling of queueing system with server breakdown has found increasing attention. Many models have been developed and successfully used in day-to-day as well as in industrial congestion problems. Numerous authors have studied the M/G/1 queue in various forms and a lot of literature is available; to cite a few, we mention Keilson and Kooharian (1960), Bhat (1964), Cohen (1957), Chaudhry and Templeton (1983). Selvam and Sivasankaran (1994) considered the two phase queueing system with server vacations. Recently, analysis of a two stage batch arrival retrial queueing model with a modified Bernoulli vacation schedule under N-policy was presented by Choudhury and Madan (2005). Choudhury and Deka (2008) studied an M/G/1 retrial queueing system with two phases of service subject to the server breakdown and repair. Artalijo and Li (2010) suggested performance analysis of a block structured discrete time retrial queue with state dependent arrivals. Pasandideh et al. (2011) gave a multi objective facility location model with bulk arrivals using two parameter-tuned meta-heuristic algorithms. Houdt (2012) did analysis of the adaptive MMAP[K]/PH[K]/1 queue by considering a multitype queue with adaptive arrivals and general impatience. An M/G/1 queue with server breakdown and multiple working vacation given by Murugan and Santhi (2015). Li and Zhang (2017) studied an M/G/1 Retrial G-Queue with General Retrial Times and Working Breakdowns.

An M/G/1 queueing system with additional optional service and no waiting capacity was studied by Madan (1994). A single server poisson input queue with a second optional channel was studied by Medhi (2002). Madan et al. (2004) proposed optional re-service for M^X /G_1, G_2/1 queue. Goswami et al. (2006) analyzed a discrete time single server bulk service queue. A batch arrival queue with a second optional service channel under N-policy was studied by Choudhury and Paul (2006). Sivasamy and Pukazhenthi (2009) proposed a discrete time bulk service queue with accessible batch. Balasubramanian et al. (2010) did the steady state analysis of a non Markovian bulk queueing system with overloading and multiple vacations. Jeyakumar and Armuganathan (2011) carried steady state analysis of a non Markovian bulk queue with restricted vacations. Jain et al. (2012) established optimal control (N, F) policy for unreliable server queue with multi-optional phase repair and startup. Chen et al. (2016) studied cost optimization of a single server queue with working breakdowns under the N policy.

Many researchers have studied how the server-breakdown affects the performance of the system and what should be the precautions taken to reduce these effects. Such type of study was fascinating and challenging for research workers of this field. Avai-Itzhak and Naor (1963) and Neuts and Lucantoni (1979) investigated some queueing problems with service station subject to breakdown. Jain (1997) considered optimal policy for single server Markovian queue with server breakdown, repair facility taking state dependent arrival rate. Gray et al. (2000) developed queueing model with service breakdown. Jain and Bhargava (2009) studied M/G/1 queueing model incorporating discouragement with Bernoulli feedback, repeated attempts, for service with modified vacation, repair in phases and unreliable server. Lin and Ke (2010) suggested genetic algorithm for optimal threshold of an infinite capacity multi-server system with triadic policy. Vemuri et al. (2011) made optimal strategic analysis of N policy in two stage M^X/M/1 queueing system with server startup and breakdown. Huang et al. (2012) presented optimum design for indoor humidity by coupling genetic algorithm with transient simulation based on contribution radio of indoor humidity and climate analysis. Liou (2015) studied markovian queue optimization analysis with an unreliable server subject to working breakdowns and impatient customers.

The matrix geometric approach can be used when the transition rate matrix has a particular block (lower or upper) Hessenberg diagonal structure of block submatrices. The model is structured as square matrix of infinite dimension that converges to finite dimension matrix using the minimal matrix to get recursive relation of probability vectors. The matrix geometric method can only be applied if the system is decomposed into two parts namely, the initial portion and the repetitive portion. Matrix geometric method (MGM) allows us to deal with the models whose activities are not necessarily exponentially distributed. It overcomes the problem of the rapid growth of the state space introduced by the need to explicitly construct the generator matrix of the underlying Markov process.Several researchers applied matrix geometric method to obtain the solution of queueing models in different frameworks. We have already reviewed some of the notable research works done by the researchers on matrix geometric method.

M/M/1 queueing system with k phases of services wherein 1^{st}phase service is essential and rest are optional services. We analyze a single server Markovian queue with breakdown under N-policy. After server breakdown, the system is unable to render service to the customers. The rest of the paper is organized as follows. In section 2,the Markovian models are described by stating the requisite assumptions and notations being used in the formulation of the mathematical model. In section 3, we define the mathematical model in different states. In section 4, the governing equations and matrix-geometric solution are provided. Various performance measures are established in scheme 5. Numerical results are provided by taking illustrations which are performed in section 6. In section 7, we conclude our investigation by highlighting the future scope of the work done.

2. Model Description

Here, we consider queueing system with k phases of service where 1^{st} phase service is an essential service (ES) and other (k-1) phase services are optional services (OS). When the first essential service of a customer is completed then he may go for first phase of optional service with probability r_0 else with probability $\overline{r_0}$ he may leave the system. After taking 2^{nd} phase of optional service with probability r_1, the customer may leave the system with rate $\overline{r_1}$ or may move to take the the next phase service. Similary the customer may go up to k^{th} optional phase service with probability r_{k-1}, or else leave the system with rate $\overline{r_{k-1}}$ for the next customer to join for the first essential service.

The basic assumptions governing the model are stated below:

- We define the state of the system at any instant by (i, j, n); i=0, 1, 2, …, k ; j = 0, 1 ; n= 0,1,2,…; where i represents the phase of the servers, while j describes the state of the service station and n denotes the number of customers.
- For convenience, we define the service station's status as follows:

$$j=\begin{cases}0; & \textit{Server is busy}\\ 1; & \textit{Server is breakdown}\end{cases}$$

- After the server breakdown, the system is unable to render service to the customers.
- There is a facility to restore the breakdown server by the repairman.
- Under N policy, the server starts providing service only when N customers are accumulated in the system. The service is henceforth provided until the system becomes empty.
- The state dependent arrival rate of the customer is given by

$$\lambda(i)=\begin{cases}\lambda\,, & \textit{when service station is busy},\ i=0,1,\ldots,k-1\\ \lambda', & \textit{when service station is turned on but under repair},\ j=0,1,\ldots,k-1\end{cases}$$

3. Mathematical Formulation of the Model

We denote the steady state probabilities for the system state by $P_{i,n}^{j}$ where n, i & j bear the same meaning as defined before. The possible states are enumerated as below:

P_N Probability of N (N ≥ 0) customers being in the system when the server is idle.

$P_{0,n}^{j}$ Probability of n customers being in the system excluding the one who is receiving the 1st essential service.

$P_{i,n}^{j}$ Probability of n customers being in the system excluding the one who is receiving the 1st optional service.

$P_{i,n}^{j}$ Probability of n customers being in the system excluding the one who is receiving the kth phase optional service, i=1,2,…,k.

The state transition diagram for different states is depicted in fig.1. To determine the steady state probabilities, we construct the governing equations using the birth and death process as follows:

$$-\lambda P_0+\overline{r_0}\mu_0\,P_{0,1}^{0}+\overline{r_1}\mu_1\,P_{1,1}^{0}+\overline{r_2}\mu_2\,P_{2,1}^{0}+\mu_3\,P_{3,1}^{0}=0 \quad \ldots(1)$$

$$-\lambda\,Pn+\lambda P_{n-1}=0, \qquad 1\le n\le N-1 \quad \ldots(2)$$

$$-(\lambda_0+\overline{r_0}\mu_0+\alpha_0+r_0\mu_0)P_{0,1}^{0}+\beta_0P_{0,1}^{1}+\mu_0P_{0,2}^{0}+\overline{r_1}\mu_1P_{1,2}^{0}+\overline{r_2}\mu_2P_{2,2}^{0}+\mu_3P_{3,2}^{0}=0 \quad \ldots(3)$$

$$-(\lambda_{k-1}+\overline{r_{k-1}}\mu_{k-1}+\alpha_{k-1}+r_{k-1}\mu_{k-1})P_{k-1,1}^{0}+\beta_{k-1}P_{k=1,1}^{1}+\mu_{k-1}P_{k-1,2}^{0}$$
$$+r_{k-2}\mu_{k-2}P_{k-2,2}^{0}=0, \qquad k>1 \quad \ldots(4)$$

$$-(\lambda_k+\alpha_k+\mu_k)P_{k,1}^{0}+\beta_kP_{k,1}^{1}+r_{k-1}\mu_{k-1}P_{k-1,1}^{0}=0 \quad \ldots(5)$$

$$-(\lambda'_{k-1}+\beta_{k-1})P_{k-1,1}^{1}+\alpha_{k-1}P_{k-1,1}^{0}=0, \qquad k\ge 1 \quad \ldots(6)$$

$$-(\lambda_0+\mu_0+\alpha_0+r_0\mu_0)P_{0,n}^{0}+\lambda_0P_{0,n-1}^{0}+\beta_0P_{0,n}^{1}+\mu_3P_{0,n+1}^{0}+\overline{r_1}\mu_1P_{1,n+1}^{0}$$
$$+\overline{r_2}\mu_2P_{2,n+1}^{0}+\mu_3P_{3,n+1}^{0}=0, \qquad 2\le n\le 4 \quad \ldots(7)$$

$$-(\lambda_{k-1}+\mu_{k-1}+\alpha_{k-1}+r_{k-1}\mu_{k-1}+\overline{r_{k-1}}\mu_{k-1})P_{k-1,n}^{0}+\lambda_{k-1}P_{k-1,n-1}^{0}$$
$$+\beta_{k-1}P_{k-1,n}^{1}+\mu_{k-1}P_{k-1,n+1}^{0}=0, \qquad k>1,\quad 2\le n\le 4 \quad \ldots(8)$$

$$-(\lambda_k+\alpha_k+\mu_k)P_{k,n}^{0}+\lambda_kP_{k,n-1}^{1}+\beta_kP_{k,n}^{1}+r_{k-1}\mu_{k-1}P_{k-1,n}^{0}=0, \quad 2\le n\le 4 \quad \ldots(9)$$

$$-(\lambda'_{k-1}+\beta_{k-1})P_{k-1,n}^{1}+\lambda'_{k-1}P_{k-1,n-1}^{1}+\alpha_{k-1}P_{k-1,n}^{0}=0, \qquad 2\le n\le 4,\ k\ge 1 \quad \ldots(10)$$

$$-(\lambda_0+\mu_0+\alpha_0+r_0\mu_0)P^0_{0,5}+\lambda_0P^0_{0,4}+\beta_0P^1_{0,5}+\mu_0P^0_{0,6}+\lambda_0P_4+\overline{r_1}\mu_1P^0_{1,6}$$
$$+\overline{r_2}\mu_2P^0_{2,6}+\mu_3P^0_{3,6}=0, \qquad n=5, k>1 \qquad \ldots(11)$$

$$-(\lambda_{k-1}+\mu_{k-1}+\alpha_{k-1}+r_{k-1}\mu_{k-1}+\overline{r_{k-1}}\mu_{k-1})P^0_{k-1,5}+\lambda_{k-1}P^0_{k-1,4}$$
$$+\beta_{k-1}P^1_{k-1,5}+\mu_{k-1}P^0_{k-1,5}=0, \qquad k>1 \qquad \ldots(12)$$

$$-(\lambda_k+\alpha_k+\mu_k)P^0_{k,5}+\lambda_kP^1_{k,4}+\beta_kP^1_{k,5}+r_{k-1}\mu_{k-1}P^0_{k-1,5}=0 \qquad \ldots(13)$$

$$-(\lambda'_{k-1}+\beta_{k-1})P^1_{k-1,5}+\lambda'_{k-1}P^1_{k-1,4}+\alpha_{k-1}P^0_{k-1,5}=0, \qquad 2\le n\le 4,\ k\ge 1 \qquad \ldots(14)$$

$$-(\lambda_0+\mu_0+\alpha_0+r_0\mu_0)P^0_{0,n}+\lambda_0P^0_{0,n-1}+\beta_0P^1_{0,n}+\mu_0P^0_{0,n+1}+\overline{r_1}\mu_1P^0_{1,n+1}$$
$$+\overline{r_2}\mu_2P^0_{2,n+1}+\mu_3P^0_{3,n+1}=0, \qquad n>5 \qquad \ldots(15)$$

$$-(\lambda_{k-1}+\mu_{k-1}+\alpha_{k-1}+r_{k-1}\mu_{k-1}+\overline{r_{k-1}}\mu_{k-1})P^0_{k-1,n}+\lambda_{k-1}P^0_{k-1,n-1}$$
$$+\beta_{k-1}P^1_{k-1,n}+\mu_{k-1}P^0_{k-1,n+1}+r_{k-2}\mu_{k-2}P^0_{k-1,n+1}=0, \qquad k>1,\quad n>5 \qquad \ldots(16)$$

$$-(\alpha_k+\mu_k)P^0_{k,n}+\lambda_kP^1_{k,n-1}+\beta_kP^1_{k,n+1}+r_{k-1}\mu_{k-1}P^0_{k-1,n}=0 \qquad \ldots(17)$$

$$-(\lambda'_{k-1}+\beta_{k-1})P^1_{k-1,n}+\lambda'_{k-1}P^1_{k-1,n-1}+\alpha_{k-1}P^0_{k-1,n}=0, \qquad n>5 \qquad \ldots(18)$$

4. The Steady State Probability Vector

Neuts (1981) developed the matrix geometric method (MGM) to solve the stationary state probabilities for the vector state Markov process with repetitive structure. Let Q be the infinitesimal generator of the continuous time Markov chain and can be given in partition form which is required for using matrix geometric approach as given below:

$$Q=\begin{bmatrix} B_0 & 0 & 0 & 0 & 0 & 0 & C_0 & 0 & \ldots & \ldots \\ A_0 & B_1 & C_1 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots \\ 0 & A_1 & B_2 & C_1 & 0 & 0 & 0 & 0 & \ldots & \ldots \\ 0 & 0 & A_1 & B_2 & C_1 & 0 & 0 & 0 & \ldots & \ldots \\ 0 & 0 & 0 & A_1 & B_2 & C_1 & 0 & 0 & \ldots & \ldots \\ 0 & 0 & 0 & 0 & A_1 & B_2 & C_1 & 0 & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & A_1 & B_2 & C_1 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \qquad \ldots(19)$$

The off-diagonal elements of Q are non-negative. The diagonal elements are strictly negative and the row sums of Q are equal to 0.

Here each capital letter represents a block sub-matrix as given below

$$B_0 = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & 0 & 0 \\ 0 & 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & 0 & -\lambda \end{bmatrix} \quad A_0 = \begin{bmatrix} \bar{r}_0\mu_0 & 0 & \bar{r}_1\mu_1 & 0 & \bar{r}_2\mu_2 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} \mu_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{r}_1\mu_1 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{r}_2\mu_2 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$B_1 =$

$$\begin{bmatrix} -(\lambda_0+r_0\mu_0+\bar{r}_0\mu_0+\alpha_0) & \alpha_0 & r_0\mu_0 & 0 & 0 & 0 & 0 & 0 \\ \beta_0 & -(\lambda_0'+\beta_0) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\lambda_1+r_1\mu_1+\bar{r}_1\mu_1+\alpha_1) & \alpha_1 & r_1\mu_1 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & -(\lambda_1'+\beta_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\lambda_2+r_2\mu_2+\bar{r}_2\mu_2+\alpha_2) & \alpha_2 & r_2\mu_2 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 & -(\lambda_2'+\beta_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda_3+\alpha_3+\mu_3) & \alpha_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_3 & 0 \end{bmatrix}$$

$B_2 =$

$$\begin{bmatrix} -(\lambda_0+\mu_0+\alpha_0+r_0\mu_0) & \alpha_0 & r_0\mu_0 & 0 & 0 & 0 & 0 & 0 \\ \beta_0 & -(\lambda_0'+\beta_0) & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\lambda_1+\mu_1+\alpha_1+r_1\mu_1) & \alpha_1 & r_1\mu_1 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & -(\lambda_1'+\beta_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\lambda_2+\mu_2+\alpha_2+r_2\mu_2) & \alpha_2 & r_2\mu_2 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 & -(\lambda_2'+\beta_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda_3+\mu_3+\alpha_3) & \alpha_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_3 & -(\lambda_3'+\beta_3) \end{bmatrix}$$

$C_1 = Diag\lfloor \lambda_0, \lambda_0', \lambda_1, \lambda_1', \lambda_2, \lambda_2', \lambda_3, \lambda_3' \rfloor$

Let X be the vector of the steady-state probabilities associated with Q, such that

X.Q=0 …(20)

and normalizing condition is

X.e=1 …(21)

Here e is a column vector of appropriate dimension with all the elements equal to 1.
We partition X conformably with the blocks of matrix Q as

$X=(X_0, X_i)$

where

$X_0= [\, X_{i,j} \,]$, i=1,…,5 j=1,…,5

$X_i=[X_{m,n}]$, m=2,3,… n=1,2,…,8

Now the governing matrix equations are given by

$X_0 B_0+X_1A_0=0$ …(22)

$X_1B_1+X_2A_1=0$ …(23)

$X_{i-1}C_1+X_iB_2+X_{i+1}A_1=0$, $1 \le i \le 4$ …(24)

$X_0C_0+X_5C_1+X_6B_2+X_7A_1=0$ …(25)

$X_{i-1}C_1+X_iB_2+X_{i+1}A_1=0,\qquad i\geq 6$ …(26)

By manipulating the above equations it can be shown that X_i can be expressed in terms of X_1 as follows:

$$X_i = X_1R^{i-1},\qquad i\geq 1 \qquad \text{…(27)}$$

$$or\ \ X_i = X_{i-1}R,\qquad i\geq 1 \qquad \text{…(28)}$$

where R is the unique minimal non-negative solution to the non-linear matrix equation

$C_1+RB_2+R^2A_1=0$ …(29)

Here, R is obtained by using equations (5.22) and (5.25). We compute matrix R by the successive substitution in the following recurrence relation:

$R(n+1)= -C_1B_2^{-1} -R^2(n)A_1B_2^{-1}=0,\quad$ for $n\geq 0$; with $R(0)=0$ …(30)

The equations (5.22) and (5.26) can be written in the matrix form as follows

$$(X_0,X_1,X_2,X_3,X_4,X_5,X_6)\begin{bmatrix} B_0 & 0 & 0 & 0 & 0 & 0 & C_0 \\ A_0 & B_1 & C_1 & 0 & 0 & 0 & 0 \\ 0 & A_1 & B_2 & C_1 & 0 & 0 & 0 \\ 0 & 0 & A_1 & B_2 & C_1 & 0 & 0 \\ 0 & 0 & 0 & A_1 & B_2 & C_1 & 0 \\ 0 & 0 & 0 & 0 & A_1 & B_2 & C_1 \\ 0 & 0 & 0 & 0 & 0 & A_1 & B_2+RA_1 \end{bmatrix}=0 \qquad \text{…(31)}$$

where X_2 is obtained by using the relation

$X_2= X_1 R$.

By using the normalizing condition, we determine

$X_0e+X_1(I-R)^{-1}e=1$ …(32)

where I is a column matrix of suitable dimension having all elements 1.

Equation (32) gives a unique solution for $[X_0,X_1,X_2,X_3,X_4,X_5,X_6]$

5. Performance Measures

Many interesting performance indices can be computed using the steady state probabilities. After obtaining probability vector in previous section, we can establish various performance measures as follows:

- The average number of customers in the system when the server is busy in providing first essential service, is

$$E(B)=\sum_{n=1}^{\infty} nP_{n,1}^{0} \qquad \text{…(33)}$$

- The average number of customers in the system when the server is broken down while busy in providing first essential service, is

$$E(D)=\sum_{n=1}^{\infty} nP_{n,1}^{1} \qquad \text{…(34)}$$

- The average number of customers in the system when the server is busy in providing 1st optional service, is

$$O(B)=\sum_{n=1}^{\infty} nP_{n,2}^{0} \qquad \text{…(35)}$$

- The average number of customers in the system when the server is broken down while busy in providing 1st optional service, is

$$O(D)=\sum_{n=1}^{\infty} n P_{n,2}^{1} \quad \ldots(36)$$

- The average number of customers in the system when the server is busy in providing 2nd optional service, is

$$O(B_1)=\sum_{n=1}^{\infty} n P_{n,3}^{0} \quad \ldots(37)$$

- The average number of customers in the system when the server is broken down while busy in providing 2nd optional service, is

$$O(D_1)=\sum_{n=1}^{\infty} n P_{n,3}^{1} \quad \ldots(38)$$

- The average number of customers in the system is given by

□D□D□
□D□ …(39)

6. Numerical Illustration

In this chapter, we discuss the numerical tractability of our model for evaluating the stationary distribution of the system size and other performance measures by taking an illustration. The steady state probabilities have been computed by coding a program in MATLAB software based on matrix-geometric method.

$$B_0=\begin{bmatrix} -.5 & .5 & 0 & 0 & 0 \\ 0 & -.5 & .5 & 0 & 0 \\ 0 & 0 & -.5 & .5 & 0 \\ 0 & 0 & 0 & -.5 & 0 \\ 0 & 0 & 0 & 0 & -.5 \end{bmatrix} \quad A_1=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .75 & 0 & 0 & 0 & 2.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_1=\begin{bmatrix} -1.52 & .02 & .40 & 0 & 0 & 0 & 0 & 0 \\ .10 & -.30 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3.46 & .06 & 1.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.50 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.37 & .07 & 1.75 & 0 \\ 0 & 0 & 0 & 0 & .30 & -.70 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3.95 & .05 \\ 0 & 0 & 0 & 0 & 0 & 0 & .30 & -.80 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} -2.02 & .02 & .40 & 0 & 0 & 0 & 0 & 0 \\ .10 & -.30 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6.46 & .06 & 1.50 & 0 & 0 & 0 \\ 0 & 0 & .20 & -.50 & -5.87 & .07 & 1.75 & 0 \\ 0 & 0 & 0 & 0 & .30 & -.07 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3.95 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .30 & .05 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.80 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.50 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} .5 & 0 & 1.2 & 0 & .75 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C_1=[0.6, 0.2, 0.7, 0.3, 0.8, 0.4, 0.9, 0.5]

Here, B_0, A_0, B_1, C_0 are rectangular matrices associated with the boundary states and A_1, B_2, C_1 are square matrices associated with the repeating states.

Now, the non-negative square matrix R is obtained from equation (5.27) as

$$R = \begin{bmatrix} 0.4033 & 0.0269 & 0.0336 & 0.0040 & 0.0121 & 0.0012 & 0.0054 & 0.0003 \\ 0.1110 & 0.6741 & 0.0138 & 0.0017 & 0.0067 & 0.0007 & 0.0030 & 0.0002 \\ 0.0398 & 0.0027 & 0.1190 & 0.0143 & 0.0350 & 0.0035 & 0.0156 & 0.0010 \\ 0.0736 & 0.0049 & 0.0370 & 0.6044 & 0.0155 & 0.0016 & 0.0069 & 0.0004 \\ 0.0808 & 0.0054 & 0.0079 & 0.0009 & 0.1496 & 0.0150 & 0.0666 & 0.0042 \\ 0.1162 & 0.0077 & 0.0140 & 0.0017 & 0.0491 & 0.5763 & 0.0219 & 0.0014 \\ 0.1353 & 0.0090 & 0.0129 & 0.0015 & 0.0052 & 0.0005 & 0.2312 & 0.0145 \\ 0.1993 & 0.0133 & 0.0246 & 0.0029 & 0.0119 & 0.0012 & 0.0530 & 0.628 \end{bmatrix}$$

With the help of these probabilities, we can calculate various performance measures using equations (33)-(38) as:

E(B)=0.3008; E(D)=0.6343; E(N)=.06754.

7. Sensitivity Analysis

The numerical results are computed by employing matrix-geometric method in order to analyze the performance indices for the M/M/1 queueing system with k phases. The effects of □□□□□□□□□□□□□□□□□□□and□□□□□□□□□□□□□□□on various performance indices are shown in figs (2-3). In figures 2 (a-b), we see that the expected number of customers initially increases sharply and then gradually as arrival rates □□and □□□□increase for different values of joining probability r_0. Figures 2(c-d) illustrate the effect of arrival rates □□□and□□□□□□on the expected number of customers in the system; it can be seen that initially there is a sharp increment in E(N) but later for higher values of □□□□□and□□□□□□□it converges to a fixed value.

The increasing trend is observed for E(N) when plotted against□□□□□□□□□□□and□□□□□respectively, for different values of r_0 as depicted in figures 3(a-c). The overall effect of the breakdown rates is that the average queue length increases with the increase in □□□□□□□□and□□□□□□□but it converges to a fixed value later. A sharp increase in E(N) can be seen in case of essential service as compared to optional service for different values of r_0 but there is a gradual increase in E(N) for higher values of breakdown rates. From figures 2 and 3, it is evident that the value of E(N) is higher for larger value of r_0, which is quite obvious. Finally, we conclude that the average queue length increases with the increase in arrival rate and breakdown rate, which is what we expect in real congestion situations.

8. Conclusion

In this chapter, we have studied M/ M /1 queueing model with multi phase of service where I^{st} phase is an essential service (ES) and remaining phases are as optional service. The server is subject to breakdown in each phase of service. With the automation of machinery system wherein server is subject to breakdown, there is a need to develop a more realistic model which can tackle congestion situations of real time systems. We have obtained the probability vectors, which are further used to determine various performance measures. The numerical results provided demonstrate the computational tractability of the analytical results as well as give an insight that how the grade of service (GoS) of the system can be ensured by the appropriate choice of repair facility.

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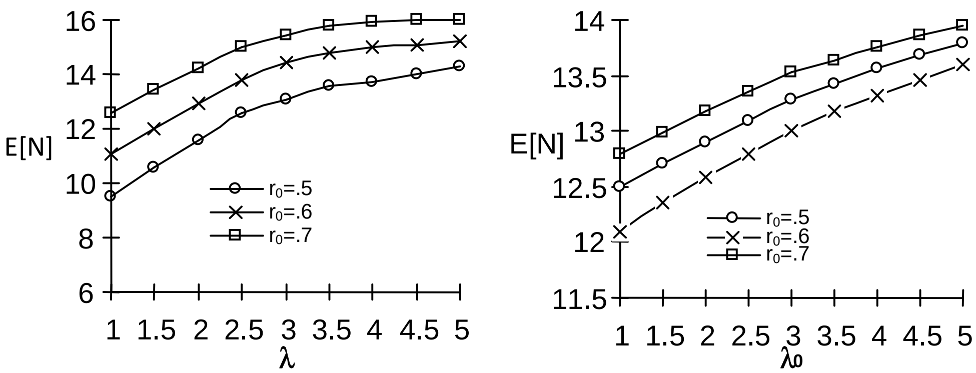
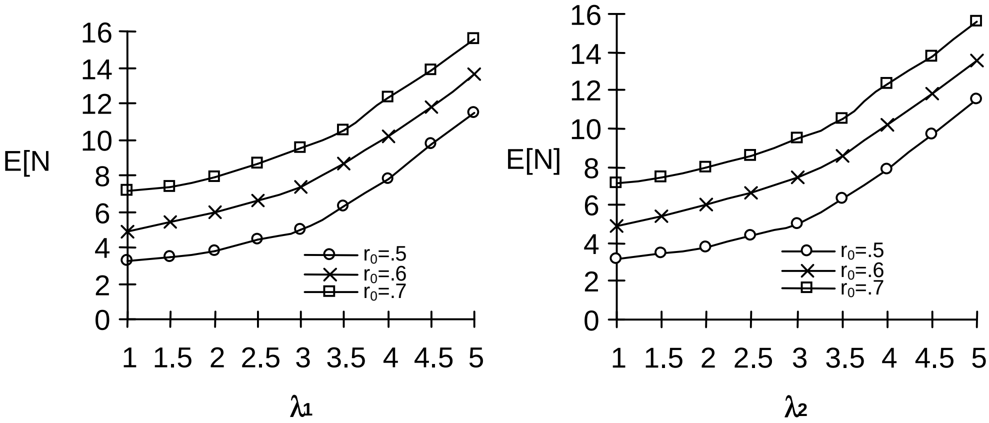
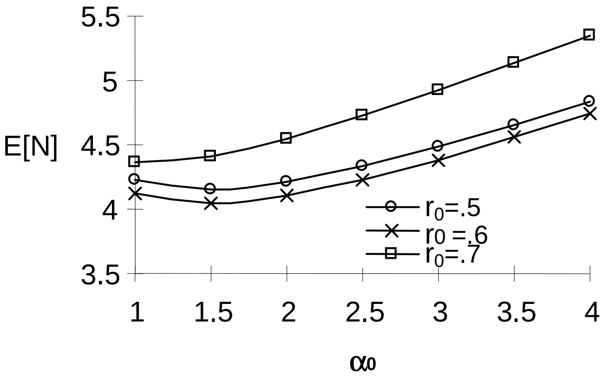


(a) (b)

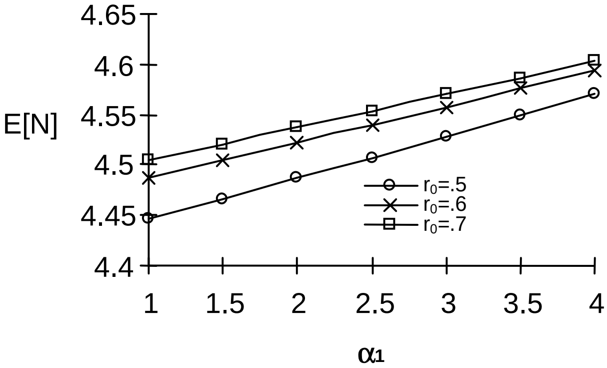


(c) (d)

Fig. 2: The expected queue length E[N] for different values of joiningprobability r_0 by varying arrival rates (a)
□□□**b**□□□□□□**c**□□□□□□**d**□□□□□□□□□



(a)



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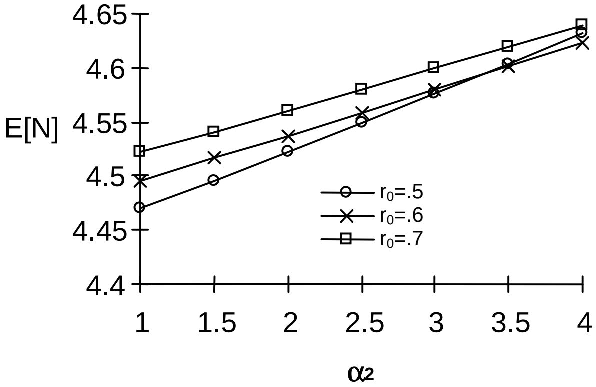

(c)

Fig. 3: The expected queue length E[N] for different values of joiningprobability r_0 by varying arrival rates □a□□□□□□□b□□□□□□c□□□□□□□□□