Even Vertex Gracefulness of Book $B_n$ when $n$ is Odd

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Abstract

Labeling of a graph $G$ is an assignment of integers either to the vertices of $G$ or edges of $G$ or both subject to certain conditions. The labeling is considered as an Injective map either from set of vertices of $G$ to a set of integers or from set of edges of $G$ to a set of integers. A graph is Even vertex graceful if there exists an injective map $f : E(G) \to \{1, 2, \ldots, 2q\}$ so that the induced map $f^+ : V(G) \to \{0, 2, 4, \ldots, 2k-2\}$ defined by $f^+(x) = \sum f(xy) \pmod{2k}$ where the sum runs over all edges $xy$ through $y$ and $k = \max(p, q)$ be such that all vertices get distinct labels. In this paper it is proven that Book $B_n$ is even vertex graceful when $n$ is odd.

KEYWORDS – Book, Even vertex gracefulness, Induced Vertex labeling, Labeling of graphs.

Introduction

Let $G = (V, E)$ be a simple graph with a finite non empty set $V$ of $p$ vertices together with set $E$ of $q$ unordered pairs of distinct points of $V$. Each pair $e = (u_1, u_2)$ of points in $E$ is an edge of $G$. A graph with $p$ vertices and $q$ edges is called a $(p, q)$ graph. A graph is said to be of order $p$. [4]

Definition : A map $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is called a graceful labeling if $f$ is one – to – one and the edges receive all the labels from $1$ to $q$ where the label of an edge is the absolute value of the difference between vertex labels at its ends. A graph having a graceful labeling is called a graceful graph. [2]

Definition : A graph is Even vertex graceful if there exists an injective map $f : E(G) \to \{1, 2, \ldots, 2q\}$ so that the induced map $f^+ : V(G) \to \{0, 2, 4, \ldots, 2k-2\}$ defined by $f^+(x) = \sum f(xy) \pmod{2k}$ where the sum runs over all edges $xy$ through $y$ and $k = \max(p, q)$ gives distinct labels to all vertices in $G$. [5]

Definition: Let $G_1$ and $G_2$ be two graphs with vertex sets $V_1$ and $V_2$. Then Cartesian products of $G_1$ and $G_2$ is denoted by $G_1 \times G_2$. To define the product $G_1 \times G_2$, consider any two points $u = (u_1 , u_2)$ and $v=(v_1, v_2)$ in $V = V_1 \times V_2$. Then $u$ and $v$ are adjacent in $G_1 \times G_2$ Whenever $u_1 = v_1$ and $u_2$ adj $v_2$ or $u_2 = v_2$ and $u_1$ adj $v_1$[4]

Definition : For $n \geq 3$ the Book $B_n$ is the cartesian product $S_n \times K_2$ where $S_n$ is the star with $n$ end – vertices and $K_2$ is the complete graph with 2–vertices. [4]

The author has used the terminology and notations of Harary [4]. So, for terms not defined here and notations not explained here refer to Harary [4]

The author has also proven “Even vertex gracefulness of book $B_n$ when $n$ is even”[7]

MAIN RESULT :
**Theorem**: For an odd integer \( n \geq 3 \), book \( B_n \) is even vertex graceful.

**Proof**: Here \( B_n \) is a book with 'n' number of pages. It is an open book with \((n+1)/2\) number of pages on left handside and \((n-1)/2\) number of pages on right hand side.

Number of vertices in \( B_n = |V(B_n)| = p = 2n + 2 \).

Number of edges in \( B_n = |E(B_n)| = q = 3n + 1 \).

Let \( V(B_n) = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, A, B\} \) and \( E(B_n) = \{e_1, e_2, \ldots, e_{3n+1}\} \).

The middle edge of the first page is numbered as \( e_1 \), then lower edge by \( e_2 \) and upper edge by \( e_3 \). The first \((n - 1)/2 \) pages contain edges numbered with \( \{e_1, e_2, e_3, \ldots, e_{3n-3}\} \) and last \((n - 3)/2 \) pages contain edges numbered with \( \{e_{(3n-1)/2}, e_{(3n+1)/2}, \ldots, e_{3n-6}\} \).

The middle two pages contain edges \( \{e_{3n-5}, e_{3n-4}, \ldots, e_{3n}\} \). The middle edge of the book is numbered as \( e_{3n+1} \).

Upper end vertices of pages are numbered as \( a_1, a_2, \ldots, a_n \) and vertices at the lower end of pages are numbered as \( b_1, b_2, \ldots, b_n \) such that edges incident at \( a_i \) are \( \{e_{3i-2}, e_{3i}\} \) and edges incident at \( b_i \) are \( \{e_{3i-2}, e_{3i-1}\} \) for all \( 1 \leq i \leq n \). The vertex where all edges \( e_{3i}, (1 \leq i \leq n) \) and \( e_{3n+1} \) meet is denoted by \( A \). The vertex where all \( e_{3i-1}, (1 \leq i \leq n) \) and \( e_{3n+1} \) meet is denoted by \( B \).

Figure 1 shows numbering of Book \( B_7 \).

Let the map \( f : E(B_n) \to \{1, 2, \ldots, 6n + 2\} \) denote labeling of edges Define \( f(e_{3n+1}) = 1 \) and \( f(e_i) = 2i + 1 \) for \( 1 \leq i \leq 3n-6 \). Then the induced map \( f'(u) = \sum f(uv) \mod 6n+2 \) where the sum runs over all edges \( uv \) through \( v \) defines vertex labeling. Clearly distinct labels are used for edges \( e_1, e_2, \ldots, e_{3n-6} \) and \( e_{3n+1} \). They are \( 3, 5, 7, \ldots, 6n-11 \).
and 1 respectively. Hence induced vertex labels for \(a_1, a_2, \ldots, a_{n-2}\) are 10, 22, \ldots, 12n – 50, 12n – 38, 12n – 26 (mod 6n+2) respectively.

Similarly induced vertex labels for \(b_1, b_2, \ldots, b_{n-3}, b_{n-2}\) are 8, 20, \ldots, 12n – 52, 12n – 40, 12n – 28 (mod 6n+2) respectively.

For the labeling of remaining six edges namely \(e_{3n-5}, e_{3n-4}, e_{3n-3}, e_{3n-2}, e_{3n-1}\), and \(e_{3n}\), three different cases are considered viz: \(n \equiv 1,3,5 \pmod{6}\)

**The case :** \(n \equiv 1 \pmod{6}\)

For first three edges, that is, \(e_{3n-5}, e_{3n-4}\) and \(e_{3n-3}\) The function \(f\) is \(f(e_i) = 2i + 1\) for \(3n-5 \leq i \leq 3n-3\) and for remaining three edges, a function is defined as \(f(e_{3n-2}) = 6n + 1\; ;\)

\(f(e_{3n-1}) = 6n -3; f(e_{3n}) = 6n - 1\)

Hence induced vertex labels for \(a_{n-1}, a_n\) are 12n – 14, 12n (mod 6n + 2) respectively. Also, induced vertex labels for \(b_{n-1}, b_n\) are 12n-16, 12n-2, (mod 6n+2) respectively.

Lastly to determine induced vertex labeling for A and B. As mentioned earlier, edges incident at A are \(e_{3i}\) for \(1 \leq i \leq n\) and edge \(e_{3n+1}\)

Hence Induced vertex Label for A

\[\equiv \left[ f(e_3) + f(e_6) + \ldots + f(e_{3n-3}) \right] + f(e_{3n}) + f(e_{3n-1}) \pmod{6n+2}\]

\[\equiv \left[ 7 + 13 + 19 + \ldots + (6n-5) \right] + (6n-1) + 1 \pmod{6n+2}\]

\[\equiv 3n^2 + 4n - 1 \pmod{6n+2}\]

\[\equiv 6n \pmod{6n+2}\]

Hence Induced vertex label for A = 6n

Similarly we calculate Induced vertex labeling for B.

The edges incident at B are \(e_{3i-1}\) for \(1 \leq i \leq n\) and \(e_{3n+1}\)

Then induced vertex label for B

\[\equiv \left[ f(e_2) + f(e_5) + f(e_8) + \ldots + f(e_{3n-a}) \right] + f(e_{3n-1}) + f(e_{3n-1}) \pmod{6n+2}\]

\[\equiv \left[ 5 + 11 + 17 + \ldots + (6n-7) \right] + (6n-3) + 1 \pmod{6n+2}\]

\[\equiv 3n^2 + 2n - 1 \pmod{6n+2}\]

\[\equiv 4n \pmod{6n+2}\]

Induced vertex label for B = 4n

**The case :** \(n \equiv 3 \pmod{6}\)

The labeling for edges \(e_{3n-5}, e_{3n-4}\) and \(e_{3n-3}\) is similar to that of the case \(n \equiv 1 \pmod{6}\)
For remaining three edges, a function is slightly different and defined as

\[ f( e_{3n-2} ) = 6n - 3 ; f( e_{3n-1} ) = 6n + 1; f( e_{3n} ) = 6n - 1 \]

Hence induced vertex labels for \( a_{n-1} \), \( a_n \) are \( 12n - 14 \), \( 12n - 4 \) (mod 6n + 2) respectively.

Also induced vertex labels for \( b_{n-1} \), \( b_n \) are \( 12n - 16 \), \( 12n - 2 \) (mod 6n + 2) respectively.

One can observe that induced vertex labeling for \( b_{n-1} \) and \( a_{n-1} \) are same as that of earlier case, as the labeling of edges incident at these vertices are same as that of earlier case. Also induced vertex labeling for \( b_n \) remains same because edge labels for \( e_{3n-2} \) and \( e_{3n-1} \) are interchanged from that of earlier labeling.

Labeling of edges incident at vertex \( A \) is same as that of earlier case. Hence in this case also induced vertex label for \( A = 6n \)

Lastly to find induced vertex labeling for \( B \)

\[
\text{Induced vertex label for } B \equiv [f(e_3)+f(e_5)+\ldots+f(e_{3n-4})]+f(e_{3n-1})+f(e_{3n}) \pmod{6n+2}
\]

\[
\equiv 3n^2 + 2n + 3 \pmod{6n + 2}
\]

\[
\equiv 4n + 4 \pmod{6n + 2}
\]

Induced vertex label for \( B = 4n + 4 \)

**The case :** \( n = 5 \pmod{6} \)

In this case a function \( f \) is defined as follows:

\[ f( e_{3n-5} ) = 6n - 7, \ f( e_{3n-4} ) = 6n - 9, \ f( e_{3n-3} ) = 6n - 5, \ f( e_{3n-2} ) = 6n + 1, \ f( e_{3n-1} ) = 6n - 3, \ f( e_{3n} ) = 6n - 1 \]

Induced vertex labels for \( a_{n-1} \) and \( a_n \) are \( 12n - 12 \) and \( 12n + 1 \) (mod 6n + 2) respectively.

Also induced vertex labels for \( b_{n-1} \), \( b_n \) are \( 12n - 16, 12n - 2 \) (mod 6n + 2) respectively.

Lastly to calculate induced vertex labels for \( A \) and \( B \)

Induced vertex label of \( A \)

\[
\equiv [ f(e_3) + f(e_5) + \ldots + f(e_{3n-6})] + [ f(e_{3n-3}) + f(e_{3n})] + [f(e_{3n+1})] \pmod{6n + 2}
\]

\[
= 6n
\]

Similarly induced vertex label of \( B \)

\[
\equiv [ f(e_2) + f(e_5) + \ldots + f(e_{3n-7})] + [ f(e_{3n-4}) + f(e_{3n-1})] + f(e_{3n+1}) \pmod{6n + 2}
\]

\[
= 4n-2
\]

**Illustration :**
Figure 2 shows Even Vertex Gracefulness of Book $B_9$ (The case $n \equiv 3 \pmod{6}$)

Conclusion: Labels assigned to edges $e_{1}, e_{2}, \ldots, e_{3n-6}$ and $e_{3n+1}$ are $3, 5, 7, \ldots, 6n-11$ and 1 respectively.

Hence induced vertex labels for $a_{1}, a_{2}, \ldots, a_{n-2}$ are $10, 22, \ldots, 12n - 50, 12n - 38, 12n - 26 \pmod{6n+2}$ respectively.

Similarly induced vertex labels for $b_{1}, b_{2}, \ldots, b_{n-3}, b_{n-2}$ are $8, 20, \ldots, 12n - 52, 12n - 40, 12n - 28 \pmod{6n+2}$ respectively.

For the remaining six edges $e_{3n-5}$ to $e_{3n}$ we considered three cases. In all three cases edge labels assigned are $6n-9, 6n-7, 6n-5, 6n-3, 6n-1, 6n+1$.

Hence induced vertex labels for $a_{n-1}, a_{n}, b_{n-1}, b_{n}, A$ and $B$ in three different cases are respectively as follows:

- In the case $n \equiv 1 \pmod{6}$ vertex labels are $12n-14, 12n, 12n-6, 12n-2, 6n$ and $4n \pmod{6n+2}$

- In the case $n \equiv 3 \pmod{6}$ vertex labels are $12n-14, 12n-4, 12n-16, 12n-2, 6n$ and $4n + 4 \pmod{6n+2}$

- In the case $n \equiv 5 \pmod{6}$ vertex labels are $12n-12, 12n+1, 12n-16, 12n-2, 6n$ and $4n - 2 \pmod{6n+2}$
Therefore \( f \) and \( f^+ \) satisfy even vertex graceful labeling. Hence when \( n \) is odd book \( B_n \) is even vertex graceful.

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