Modeling with Fractional Frequency Sumudu Transform by Inverse Difference Operator

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Abstract

This article aims to define a new type of fractional frequency Sumudu transform using inverse difference operator with shift value `h' while executing the fractional frequency Sumudu transform it's a new technique using h-difference operator. Fractional frequency Sumudu transform at polynomial factorial, geometric and trignometric functional are presented. Also, the researcher develop the convolution product of fracional frequency Sumudu transform. Numerical results are verified and diagrams are generated by MATLAB .

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1. Introduction

There are several integral transforms such as the Laplace, Millen, Hankel and Fourier transforms that are used to solve differential equations which apper in many fields of science and engineering. In the early 1990's, Watugala [7, 8] introduced the Sumudu transform and applied it to solve ordinary differential equations. Watugala's work was followed by Weerakoon who introduced the complex inversion formula for the Sumudu transform [9, 10]. The fundamental properties of this transform, which is thought to be an alternative to the Laplace transform were then established in many articles [11, 12].

The Sumudu transform is defined over the set of functions

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, \left| f(t) \right| < Me^{\frac{|t|}{\tau_j}}, if t \in (-1)^j \times [0, \infty) \right\}, \tag{1}$$

by

$$F(u) = S\{f(t)\}(u) = \frac{1}{u}\int_0^\infty f(t) e^{-\frac{t}{u}} dt, u \in (-\tau_1, \tau_2). \tag{2}$$

Also, there is an bridge between Laplace transform and Sumudu transform which has many applications in applied sciences. Moreover, some properties of Sumudu transform makes it more advantageous than the Sumudu transform of a Heaviside step function is a also Heaviside step function in the transformed domain;

$-St^n = n!u^n$;

$-\lim_{t \to \infty} f(t) = \lim_{u \to 0} F(u)$;

$-\lim_{t \to 0} f(t) = \lim_{u \to 0} F(u)$; $-if c > 0, Sf(ct) = F(cu)$

Recently, it was proved that by using the Sumudu transform, one can transform the two dimentional transport equation into a Fredholm integral equation [16]. In [15], the authors applied the Sumudu transform to fractional differential equations which have many applications in the fields of science (see [17] and the references therein).

Begin with classical definition of Laplace transform an arbitrary time scales, the concept of the $h-$Laplace and consequently the discrete Laplace transformed were specified in [13]. It was initiated by Stefan Hilger [14]. This theory is a tool that unifies the theories of continuous and discrete time systems. It is the subject of recent studies on many different fields in which dynamic process can be described with discrete or continuous models. The recent applications of fractional Laplace transform using difference equation are found in [18, 19].

2. Preliminaries

In this section, we present basic theory of the $h-$difference operator Δ_h. The polynomial factorial is defined $t_h^{(m)} = t(t-h)(t-2h)\cdots(t-(m-1)h)$, $h > 0$ for non-negative integer m and using Stirling numbers of first kind s_r^m and second kind S_r^m , the relation between polynomial and polynomial factorials are given by,

$$(i)\; t_h^{(m)} = \sum_{r=1}^{m} s_r^m h^{m-r} t^r, (ii)\; t^m = \sum_{r=1}^{m} S_r^m h^{m-r} t_h^{(r)}. \tag{3}$$

Definition 2.1 *Let* $u(t), t \in [0,\infty)$, *be a real or complex valued function and* $h > 0$ *be a fixed shift value. Then, the* $h-$*difference operator* Δ_h *on* $u(t)$ *is defined as*

$$\Delta_h u(t) = \frac{u(t+h)-u(t)}{h}, \tag{4}$$

and its infinite $h-$ difference sum is defined by

$$\Delta_h^{-1} u(t) = h\sum_{r=0}^{\infty} u(t+rh), \tag{5}$$

Definition 2.2 *Let* $u(t)$ *and* $v(t)$ *are the two real valued functions defined on* $(-\infty,\infty)$ *and if* $\Delta_h v(t) = u(t),$ *then the finite inverse principle law is given by*

$$v(t) - v(t-mh) = h\sum_{r=1}^{m} u(t-rh), m \in Z^+ \tag{6}$$

Applying the Definition 2.1, we get the modified identities as follows:

$$(i)\; \Delta_h t_h^{(m)} = mt_h^{(m-1)},\; (ii)\; \Delta_h^{-1} t_h^{(m)} = \frac{t_h^{(m+1)}}{m+1} (iii)\; \Delta_h^{-1} t^m = \sum_{r=1}^{m} \frac{S_r^m h^{m-r} k_h^{(r)}}{r+1}. \tag{7}$$

Lemma 2.3 *Let* $h > 0$ *and* $u(t)$, $w(t)$ *are real valued bounded functions. Then*

$$\Delta_h^{-1}(u(t)w(t)) = u(t)\Delta_h^{-1} w(t) - \Delta_h^{-1}(\Delta_h^{-1} w(t+h)\Delta_h u(t)). \tag{8}$$

Lemma 2.4 *Let* $t \in (-\infty, \infty), h > 0$, $\tau \in R$ *and* $\nu > 0$, *then we have*

$$\Delta_h^{-1} e^{-\frac{t}{\tau^\nu}} = \frac{h e^{-\frac{t}{\tau^\nu}}}{(e^{-\frac{h}{\tau^\nu}} - 1)} \tag{9}$$

Proof. Taking $u(t) = e^{-\frac{t}{\tau^\nu}}$ in Definition 2.1, we have

$$\Delta_h e^{-\frac{t}{\tau^\nu}} = \frac{e^{\frac{1}{\tau^\nu}(t+h)} - e^{\frac{t}{\tau^\nu}}}{h} = \frac{e^{-\frac{t}{\tau^\nu}}[e^{-\frac{h}{\tau^\nu}-1}]}{h}.$$

Apply Δ_h^{-1} on both sides, we get (9).

Corollary 2.5 *Let* $t \in (-\infty, \infty), h > 0$, $\tau \in R$ *and* $\nu > 0$, *then we have*

$$\frac{h e^{-\frac{t}{\tau^\nu}}}{(e^{-\frac{h}{\tau^\nu}} - 1)} - \frac{h e^{-\frac{t-mh}{\tau^\nu}}}{(e^{-\frac{h}{\tau^\nu}} - 1)} = h \sum_{r=1}^{m} e^{-\frac{t-rh}{\tau^\nu}} \tag{10}$$

Proof. The proof follows by equating (9) and the finite inverse principle law given in (6).

Theorem 2.6 *Let* $t \in (-\infty, \infty)$, $h > 0$, $\tau \in R$, $\nu > 0$ *be shift value and* $2(\cosh\frac{h}{\tau^\nu} - \cos ph) \neq 0$,. *Then we have*

$$\Delta_h^{-1}(e^{-\frac{t}{\tau^\nu}} \cos pt) = \frac{h e^{\frac{h}{\tau^\nu}} e^{-\frac{t}{\tau^\nu}} (e^{-\frac{h}{\tau^\nu}} \cos p(t-h) - cospt)}{2\cos h\frac{h}{\tau^\nu} - 2\cos ph}, \tag{11}$$

$$\Delta_h^{-1}(e^{-\frac{t}{\tau^\nu}} \sin pt) = \frac{h e^{\frac{h}{\tau^\nu}} e^{-\frac{t}{\tau^\nu}} (e^{-\frac{h}{\tau^\nu}} \sin p(t-h) - sinpt)}{2\cos h\frac{h}{\tau^\nu} - 2\cos ph}. \tag{12}$$

Proof. Taking $u(t) = e^{-\frac{t}{\tau^\nu}}$ $v(t) = \cos pt$ in (8), we get

$$\Delta_h^{-1}(e^{-\frac{t}{\tau^\nu}} \cos pt) = \text{Re part } \Delta_h^{-1}(e^{-\frac{t}{\tau^\nu}} e^{ipt}) = \text{Re part } \Delta_h^{-1}(e^{(-\frac{1}{\tau^\nu}+ip)t}).$$

Now $\Delta_h e^{(-\frac{1}{\tau^\nu}+ip)t} = \dfrac{e^{(-\frac{1}{\tau^\nu}+ip)(t+h)} - e^{(-\frac{1}{\tau^\nu}+ip)t}}{h} = \dfrac{e^{(-\frac{1}{\tau^\nu}+ip)t}[e^{(-\frac{1}{\tau^\nu}+ip)h-1}]}{h}$.

Taking Δ_h^{-1} on both sides, we arrives

$$\Delta_h^{-1} e^{(-\frac{1}{\tau^{\nu}}+ip)t} = \frac{he^{(-\frac{1}{\tau^{\nu}}+ip)t}}{(e^{(-\frac{1}{\tau^{\nu}}+ip)h}-1)}.$$

$$\text{Re part } \Delta_h^{-1} e^{(-\frac{1}{\tau^{\nu}}+ip)t} = \text{Re part } \frac{he^{(-\frac{1}{\tau^{\nu}}+ip)t}}{(e^{(-\frac{1}{\tau^{\nu}}+ip)h}-1)} \frac{(e^{(-\frac{1}{\tau^{\nu}}+ip)h}-1)}{(e^{(-\frac{1}{\tau^{\nu}}+ip)h}-1)},$$

$$= \text{Re part } \frac{h[e^{-\frac{1}{\tau^{\nu}}(t+h)} e^{ip(t-h)} - e(-\frac{1}{\tau^{\nu}}+ip)t]}{e^{-\frac{2h}{\tau^{\nu}}} - e^{-\frac{h}{\tau^{\nu}}}(e^{iph}+e^{-iph})+1}.$$

On simplifying the above expression we get (11). Similarly we get the proof of (12).

3 RESULTS ON FRACTIONAL FREQUENCY SUMUDU TRANSFORM

In this section, we derive several results and identities on fractional frequency Sumudu transform on polynomial factorial, trignometric and geometric functions.

Definition 3.1 *Let* $u(t)$ *be the real valued function,* $h > 0$ *and* $\nu \in R^{+}$. *If* $\lim_{t\to\infty} \Delta_h^{-1} u(t) e^{-\frac{t}{\tau^{\nu}}} = 0$, *then the fractional frequency sumudu transform is defined as*

$$S_{h,\nu}(u(t)) = \frac{1}{\tau} \Delta_h^{-1} u(t) e^{-\left(\frac{t}{\tau^{\nu}}\right)} \Big|_0^{\infty} = \frac{h}{\tau} \sum_{r=0}^{\infty} u(rh) e^{-\left(\frac{rh}{\tau^{\nu}}\right)} \tag{13}$$

Theorem 3.2 *If* $\cosh \frac{h}{\tau^{\nu}} - \cos ph \neq 0$, *then we have*

$$S_{h,\nu}[\sin pt] = \frac{1}{\tau} \frac{h \sin ph}{2(\cosh \frac{h}{\tau^{\nu}} - \cos ph)} \quad \textit{and} \quad S_{h,\nu}[\cos pt] = \frac{1}{\tau} \frac{h(e^{\frac{h}{\tau^{\nu}}} - \cos ph)}{2(\cosh \frac{h}{\tau^{\nu}} - \cos ph)}. \tag{14}$$

Proof. The proof of (14) follows by multiplying $\frac{1}{\tau}$ and applying limits from 0 to ∞ in (11) and (12). The following example is the numerical verification of Theorem (3.2)

Example 3.3 *For the particular values* $\nu = 0.5, \tau = 2, p = 3$ *and* $h = 3$*, (14) is verifieed by* $MATLAB$*. The coding is given by*

$$(3.*(exp((3)./(2.\wedge(0.5))-cos(3.*3)).\,/\,(4.*(cosh((3)./(2.\wedge(0.5)))-cos(3.*3)))) =$$

$$(3./2).*symsum(cos(3.*3.*r).*exp((-r.*3)./(2.\wedge(0.5)),r,0,inf)$$

Remark 3.4 *When* $h \to 0$ *and* $\nu = 1$*, we arrives*

$$S(\sin pt) = \frac{p}{\tau^2 + p^2} \quad and \quad S(\cos pt) = \frac{\tau}{\tau^2 + p^2}.$$

Theorem 3.5 *If* $e^{-(\frac{t}{\tau^\nu} \pm p)h} \neq 1$ *and* $\tau > 0$*, then*

$$\begin{aligned} S_{h,\nu}(\sinh pt) &= \frac{h}{2\tau}\Big[\frac{1}{e^{-(\frac{t}{\tau^\nu}+p)h} - 1} + \frac{1}{1 - e^{-(\frac{t}{\tau^\nu}-p)h}}\Big], \\ S_{h,\nu}(\cosh pt) &= \frac{h}{2\tau}\Big[\frac{1}{1 - e^{-(\frac{t}{\tau^\nu}-p)h}} + \frac{1}{1 - e^{-(\frac{t}{\tau^\nu}+p)h}}\Big]. \end{aligned} \tag{15}$$

Remark 3.6 *When* $h \to 0$ *and* $\nu = 1$*, we get*

$$S(\sinh pt) = \frac{p}{\tau^2 - p^2} \quad and \quad S(\cosh pt) = \frac{\tau}{\tau^2 - p^2}.$$

Proof. From the definition of Sumudu transform, we have

$$S_{h,\nu}(\cosh pt) = \frac{1}{2\tau}\Delta_h^{-1}(e^{pt} + e^{-pt})e^{-(\frac{t}{\tau^\nu})}\Big|_0^\infty = \frac{1}{2\tau}[\Delta_h^{-1}e^{-(\frac{t}{\tau^\nu}-p)t} + \Delta_h^{-1}e^{-(\frac{t}{\tau^\nu}+p)t}].$$

Now, $\Delta_h e^{-(\frac{t}{\tau^\nu}-p)t} = \dfrac{e^{-(\frac{t}{\tau^\nu}-p)(t+h)} - e^{-(\frac{t}{\tau^\nu}-p)t}}{h} = \dfrac{e^{-(\frac{t}{\tau^\nu}-p)t}[e^{-(\frac{t}{\tau^\nu}-p)h} - 1]}{h}$.

Apply Δ_h^{-1} on both sides we get,

$$\Delta_h^{-1}e^{-(\frac{t}{\tau^\nu}-p)t} = \frac{he^{-(\frac{t}{\tau^\nu}-p)t}}{(e^{-(\frac{t}{\tau^\nu}-p)h} - 1)}$$

Similarly, $\Delta_h^{-1}e^{-(\frac{t}{\tau^\nu}+p)t} = \dfrac{he^{-(\frac{t}{\tau^\nu}+p)t}}{(e^{-(\frac{t}{\tau^\nu}+p)h} - 1)}$.

$$S_{h,\nu}(\cosh pt) = \frac{1}{2\tau}\Big[\frac{he^{-(\frac{t}{\tau^\nu}-p)t}}{(e^{-(\frac{t}{\tau^\nu}-p)h} - 1)} + \frac{he^{-(\frac{t}{\tau^\nu}+p)t}}{(e^{-(\frac{t}{\tau^\nu}+p)h} - 1)}\Big],$$

$$= \frac{h}{2\tau}\Big[-\frac{1}{(e^{-(\frac{t}{\tau^{\nu}}-p)h}-1)} - \frac{1}{(e^{-(\frac{t}{\tau^{\nu}}+p)h}-1)}\Big],$$

$$S_{h,\nu}(\cosh pt) = \frac{h}{2\tau}\Big[\frac{1}{1-e^{-(\frac{t}{\tau^{\nu}}-p)h}} + \frac{1}{1-e^{-(\frac{t}{\tau^{\nu}}+p)h}}\Big].$$

In the similar manner, we arrives

$$S_{h,\nu}(\sin hpt) = \frac{1}{2\tau}\Delta_h^{-1}(e^{pt}-e^{-pt})e^{-(\frac{t}{\tau^{\nu}})}\Big|_0^{\infty}$$

$$= \frac{1}{2\tau}\Big[\Delta_h^{-1}e^{-(\frac{t}{\tau^{\nu}}-p)t} - \Delta_h^{-1}e^{-(\frac{t}{\tau^{\nu}}+p)t}\Big].$$

$$S_{h,\nu}(\sinh pt) = \frac{1}{2\tau}\Big[\frac{he^{-(\frac{t}{\tau^{\nu}}-p)t}}{(e^{-(\frac{t}{\tau^{\nu}}-p)h}-1)} - \frac{he^{-(\frac{t}{\tau^{\nu}}+p)t}}{(e^{-(\frac{t}{\tau^{\nu}}+p)h}-1)}\Big],$$

$$= \frac{h}{2\tau}\Big[\frac{1}{(e^{-(\frac{t}{\tau^{\nu}}+p)h}-1)} - \frac{1}{(e^{-(\frac{t}{\tau^{\nu}}-p)h}-1)}\Big],$$

$$S_{h,\nu}(\sinh pt) = \frac{h}{2\tau}\Big[\frac{1}{e^{-(\frac{t}{\tau^{\nu}}+p)h}-1} + \frac{1}{1-e^{-(\frac{t}{\tau^{\nu}}-p)h}}\Big],$$

which completes the proof of (15).

Theorem 3.7 *Let* $t \in (0,\infty)$, $h > 0$ *and* $\tau > 0$, *then*

$$S_{h,\nu}(t_h^{(n)}) = \frac{h^{n+1}n!e^{\frac{h}{\tau^{\nu}}}}{\tau(e^{\frac{h}{\tau^{\nu}}}-1)^{n+1}}. \qquad (16)$$

Proof. Taking $u(t) = t_h^{(1)}$, $w(t) = e^{-\frac{t}{\tau^{\nu}}}$ in (6), we get

$$\Delta_h^{-1}t_h^{(1)}e^{-\frac{t}{\tau^{\nu}}} = t_h^{(1)}\Delta_h^{-1}e^{-\frac{t}{\tau^{\nu}}} - \Delta_h^{-1}[\Delta_h^{-1}e^{-\frac{t}{\tau^{\nu}}}(t+h)\Delta_h t_h^{(1)}]$$

$$= t_h^{(1)}\frac{he^{-\frac{t}{\tau^{\nu}}}}{(e^{-\frac{h}{\tau^{\nu}}}-1)} - \Delta_h^{-1}\Big[\frac{he^{-\frac{1}{\tau^{\nu}}(t+h)}}{(e^{-\frac{h}{\tau^{\nu}}}-1)}\Big] = t_h^{(1)}\frac{he^{-\frac{t}{\tau^{\nu}}}}{(e^{-\frac{h}{\tau^{\nu}}}-1)} - \frac{h}{(e^{-\frac{h}{\tau^{\nu}}}-1)}[\Delta_h^{-1}e^{-\frac{1}{\tau^{\nu}}(t+h)}]$$

$$\Delta_h^{-1} t_h^{(1)} e^{-\frac{t}{\tau^\nu}} = t_h^{(1)} \frac{h e^{-\frac{t}{\tau^\nu}}}{(e^{-\frac{h}{\tau^\nu}} - 1)} - \frac{h^2 e^{-\frac{1}{\tau^\nu}(t+h)}}{(e^{-\frac{h}{\tau^\nu}} - 1)^2}$$

$$\Delta_h^{-1} t_h^{(1)} e^{-\frac{t}{\tau^\nu}} \Big|_{t=0}^{\infty} = t_h^{(1)} \frac{h e^{-\frac{t}{\tau^\nu}}}{(e^{-\frac{h}{\tau^\nu}} - 1)} - \frac{h^2 e^{-\frac{1}{\tau^\nu}(t+h)}}{(e^{-\frac{h}{\tau^\nu}} - 1)^2} \Big|_{t=0}^{\infty}$$

$$= \frac{h^2 e^{-\frac{h}{\tau^\nu}}}{(e^{-\frac{h}{\tau^\nu}} - 1)^2} = \frac{h^2 e^{-\frac{h}{\tau^\nu}} e^{\frac{2h}{\tau^\nu}}}{(1 - e^{\frac{h}{\tau^\nu}})^2}$$

$$\Delta_h^{-1} t_h^{(1)} e^{-\frac{t}{\tau^\nu}} \Big|_{t=0}^{\infty} = \frac{h^2 e^{\frac{h}{\tau^\nu}}}{(e^{\frac{h}{\tau^\nu}} - 1)^2} \Rightarrow \frac{1}{\tau} \Delta_h^{-1} t_h^{(1)} e^{-\frac{t}{\tau^\nu}} \Big|_{t=0}^{\infty} = \frac{h^2 e^{\frac{h}{\tau^\nu}}}{u(e^{\frac{h}{\tau^\nu}} - 1)^2}$$

$$S_{h,\nu}(t_h^{(1)}) = \frac{h^2 e^{\frac{h}{\tau^\nu}}}{\tau(e^{\frac{h}{\tau^\nu}} - 1)^2}. \qquad (17)$$

Again taking, $u(t) = t_h^{(2)}$, $w(t) = e^{-\frac{t}{\tau^\nu}}$ in (6), which gives

$$\Delta_h^{-1}[t_h^{(2)} e^{-\frac{t}{\tau^\nu}}] = t_h^{(2)} \Delta_h^{-1} e^{-\frac{t}{\tau^\nu}} - \Delta_h^{-1}[\Delta_h^{-1} e^{-\frac{t}{\tau^\nu}} (t+h) \Delta_h t_h^{(2)}]$$

$$= t_h^{(2)} \frac{h e^{-\frac{t}{\tau^\nu}}}{(e^{-\frac{h}{\tau^\nu}} - 1)} - \Delta_h^{-1}[\frac{h e^{-\frac{1}{\tau^\nu}(t+h)}}{(e^{-\frac{h}{\tau^\nu}} - 1)} 2t_h^{(1)}]$$

$$= t_h^{(2)} \frac{h e^{-\frac{t}{\tau^\nu}}}{(e^{-\frac{h}{\tau^\nu}} - 1)} - \frac{2h}{(e^{-\frac{h}{\tau^\nu}} - 1)} \Delta_h^{-1}[t_h^{(1)} e^{-\frac{1}{\tau^\nu}(t+h)}]$$

$$\Delta_h^{-1} t_h^{(2)} e^{-\frac{t}{\tau^\nu}} = t_h^{(2)} \frac{h e^{-\frac{t}{\tau^\nu}}}{(e^{-\frac{h}{\tau^\nu}} - 1)} - \frac{2h}{(e^{-\frac{h}{\tau^\nu}} - 1)} [t_h^{(1)} \frac{h e^{-\frac{1}{\tau^\nu}(t+h)}}{(e^{-\frac{h}{\tau^\nu}} - 1)} - \frac{h^2 e^{-\frac{1}{\tau^\nu}(t+2h)}}{(e^{-\frac{h}{\tau^\nu}} - 1)^2}]$$

$$= t_h^{(2)} \frac{h e^{-\frac{t}{\tau^\nu}}}{(e^{-\frac{h}{\tau^\nu}} - 1)} - \frac{2h^2 t_h^{(1)} e^{-\frac{1}{\tau^\nu}(t+h)}}{(e^{-\frac{h}{\tau^\nu}} - 1)^2} + \frac{2h^3 e^{-\frac{1}{\tau^\nu}(t+2h)}}{(e^{-\frac{h}{\tau^\nu}} - 1)^3}$$

$$\Delta_h^{-1} t_h^{(2)} e^{-\frac{t}{\tau^\nu}} \Big|_{t=0}^{\infty} = t_h^{(2)} \frac{h e^{-\frac{t}{\tau^\nu}}}{(e^{-\frac{h}{\tau^\nu}} - 1)} - \frac{2h^2 t_h^{(1)} e^{-\frac{1}{\tau^\nu}(t+h)}}{(e^{-\frac{h}{\tau^\nu}} - 1)^2} + \frac{2h^3 e^{-\frac{1}{\tau^\nu}(t+2h)}}{(e^{-\frac{h}{\tau^\nu}} - 1)^3} \Big|_{t=0}^{\infty}$$

$$\Delta_h^{-1} t_h^{(2)} e^{-\frac{t}{\tau^\nu}} \Big|_{t=0}^{\infty} = -\frac{2h^3 e^{-\frac{2h}{\tau^\nu}}}{\left(e^{-\frac{h}{\tau^\nu}}-1\right)^3} = -\frac{2h^3 e^{-\frac{2h}{\tau^\nu}} e^{\frac{3h}{\tau^\nu}}}{\left(1-e^{\frac{h}{\tau^\nu}}\right)^3}$$

$$\Delta_h^{-1} t_h^{(2)} e^{-\frac{t}{\tau^\nu}} \Big|_{t=0}^{\infty} = \frac{2h^3 e^{\frac{h}{\tau^\nu}}}{\left(e^{\frac{h}{\tau^\nu}}-1\right)^3} \Rightarrow \frac{1}{\tau} \Delta_h^{-1} t_h^{(2)} e^{-\frac{t}{\tau^\nu}} \Big|_{t=0}^{\infty} = \frac{2h^3 e^{\frac{h}{\tau^\nu}}}{\tau\left(e^{\frac{h}{\tau^\nu}}-1\right)^3}$$

$$S_{h,\nu}(t_h^{(2)}) = \frac{2h^3 e^{\frac{h}{\tau^\nu}}}{\tau\left(e^{\frac{h}{\tau^\nu}}-1\right)^3}$$

In general, induction on $'n'$ we get (16).

Corollary 3.8 *Let* $t \in (0,\infty)$, $h>0$ *and* $\tau>0$, *then* $S_{h,\nu}(t^n) = \sum_{r=0}^{n} \frac{S_r^n h^{n+1} e^{\frac{h}{\tau^\nu}}}{(e^{h/\tau^\nu}-1)^{n+1}}$

Proof. The proof follows from (ii) of (3), (ii) of (7) and Theorem 3.7,

$$\Delta_h^{-1} t^n e^{-\frac{t}{\tau^\nu}} = \Delta_h^{-1}\Big[\sum_{r=0}^{n} S_r^n h^{n-r} t_h^{(r)}\Big] e^{\frac{h}{\tau^\nu}} = \sum_{r=0}^{n} S_r^n h^{n-r} \Big[\Delta_h^{-1} t_h^{(n)} e^{\frac{h}{\tau^\nu}}\Big]$$

$$= \sum_{r=0}^{n} S_r^n h^{n-r} \left[\frac{h^{n+1} n! e^{\frac{h}{\tau^\nu}}}{\left(e^{\frac{h}{\tau^\nu}}-1\right)^{n+1}}\right]$$

$$S_{h,\nu}(t^n) = \sum_{r=0}^{n} \frac{S_r^n h^{n+1} e^{\frac{h}{\tau^\nu}}}{\left(e^{\frac{h}{\tau^\nu}}-1\right)^{n+1}}.$$

The following example is the numerical verification of Theorem 3.7.

Example 3.9 *Taking* $n=2$ *in Theorem (16), we obtain*

$$S_{h,\nu}(t_h^{(2)}) = \frac{2h^3 e^{\frac{h}{\tau^\nu}}}{\tau\left(e^{\frac{h}{\tau^\nu}}-1\right)^3} = \frac{h}{\tau}\sum_{r=0}^{\infty} (rh)_2^{(2)} e^{-\frac{rh}{\tau^\nu}},$$

which verified for the values $h=2, \tau=3$ and $\nu=0.4$ by MATLAB coding given below:

$$(16.*exp(2./3.\wedge(0.4)))./(3.*(exp(2./3.\wedge(0.4))-1).\wedge 3) =$$
$$(2./3).*symsum(4.*r.*(r-1).*exp((-r.*2)./(3.\wedge(0.4))), r, 0, inf)$$

The following Figure.1 is the input function(signal) as polynomial factorial for the time factor t and Figure.2 is the fractional generalized Sumudu transform in the frequency domain τ and also here in the frequency domain the fraction ν varies as $0.4, 0.3, 0.2, 0.1$ which are generated by MATLAB are shown below.

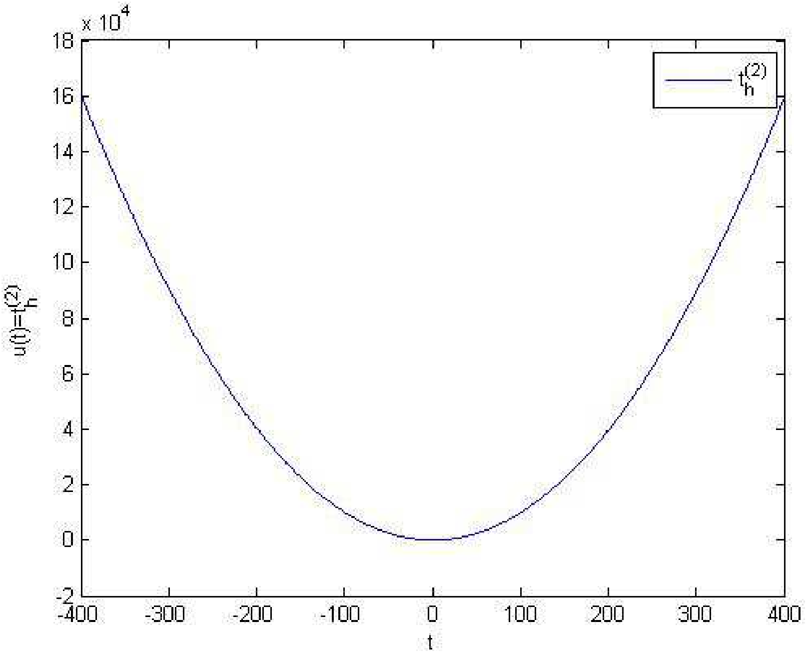

Figure 1: Time(t)

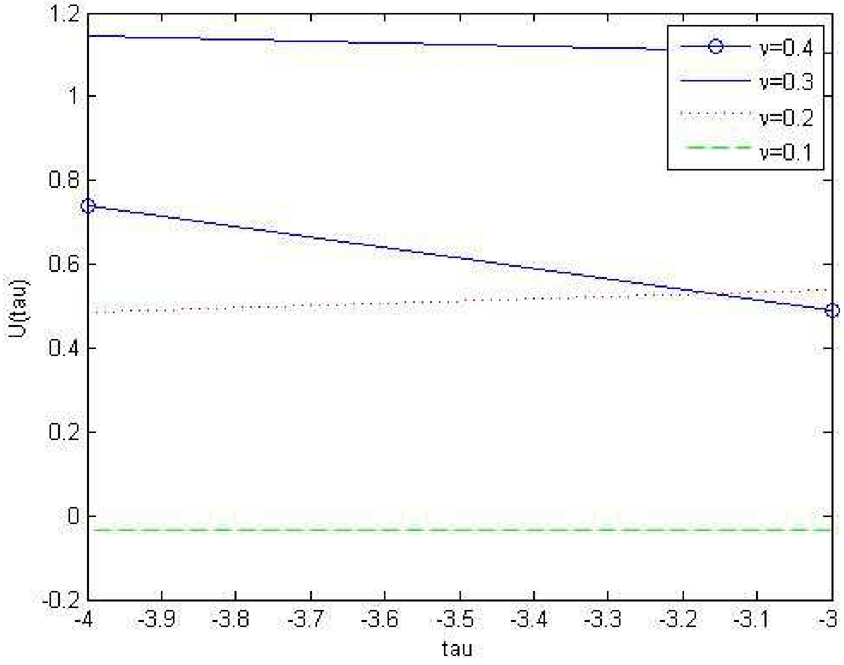

Figure 2: Frequency (τ)

4. Convolution Product and Fractional Sumudu Transforms

In this section, we defined convolution product with Fractional Sumudu transforms. The following definitions are motivated using $h-$difference operator.

Definition 4.1 *Let* $u(t)$ *be the real valued function, then the incomplete generalized Sumudu transform is defined by*

$$S_h[u(t),b] = \Delta_h^{-1}u(t)e^{-\tau t}\Big|_0^b \tag{18}$$

Definition 4.2 *Let* $u(t)$ *and* $v(t)$ *are the two real valued functions, then the convolution product is defined by*

$$(u\circ v)(t) = \Delta_h^{-1}u(\xi-t)v(\xi)\Big|_{\xi=t}^{\infty}, \quad t>0 \tag{19}$$

The following lemma shows that the relation between convolution product and Fractional Sumudu transform.

Lemma 4.3 *Let* $\mu\in R^{+}$, $u(t)$ *and* $v(t)$ *are the real valued functions, then*

(i) $u\circ e^{-\frac{1}{\mu^{\nu}}} = S_{h,\nu}[u]\cdot e^{-\frac{1}{\mu^{\nu}}}$ (ii) $S_{h,\nu}[u\circ v] = S_{h,\nu}[S_h(u(t_1),\xi)]$

Proof. (i) From (19), we get $(u\circ e^{-\frac{1}{\mu^{\nu}}})(t) = \Delta_h^{-1}u(\xi-t)e^{-\frac{1}{\mu^{\nu}}h}\Big|_{\xi=t}^{\infty}$

Taking $t_1=\xi-t$, $(u\circ e^{-\frac{1}{\mu^{\nu}}})(t) = \Delta_h^{-1}u(t_1)e^{-\frac{1}{\mu^{\nu}}(t_1+t)}\Big|_{t_1=0}^{\infty}$

Then, we have

$$(u\circ e^{-\frac{1}{\mu^{\nu}}})(t) = e^{-\frac{1}{\mu^{\nu}}(t)}\Delta_h^{-1}u(t_1)e^{-\frac{1}{\mu^{\nu}}(t_1)}\Big|_{t_1=0}^{\infty},$$

$$(u\circ e^{-\frac{1}{\mu^{\nu}}})(t) = e^{-\frac{1}{\mu^{\nu}}(t)}S_{h,\nu}[u],$$

$$u\circ e^{-\frac{1}{\mu^{\nu}}} = S_{h,\nu}[u]\cdot e^{-\frac{1}{\mu^{\nu}}(t)}.$$

(ii) Now, $S_{h,\nu}[u\circ v] = \Delta_h^{-1}(u\circ v)(t)e^{-\frac{1}{\mu^{\nu}}(t)}\Big|_{t=0}^{\infty} = \Delta_h^{-1}[\Delta_h^{-1}u(\xi-t)v(\xi)\Big|_{\xi=t}^{\infty}]e^{-\frac{1}{\mu^{\nu}}(t)}\Big|_{t=0}^{\infty}$

Now applying Fubini's Theorem, we get

$$\Delta_h^{-1}e^{-\frac{1}{\mu^{\nu}}(t)}[\Delta_h^{-1}u(\xi-t)v(\xi)\Big|_{\xi=t}^{\infty}]\Big|_{t=0}^{\infty} = \Delta_h^{-1}v(\xi)[\Delta_h^{-1}u(\xi-t)e^{-\frac{1}{\mu^{\nu}}(t)}\Big|_{t=0}^{\infty}]\Big|_{\xi=0}^{\infty}.$$

$$S_{h,\nu}[u\circ v] = S_{h,\nu}[S_h(u(t_1),\xi)].$$

The following example is the analysis of the convolution product both numerically and diagrams are generated by MATLAB.

Example 4.4 *Consider the following functions*

$$u(t) = \begin{cases} e^{-\frac{t}{\tau^{\nu}}}, & t \in (0,\infty) \\ 0, & \text{otherwise} \end{cases} \qquad v(t) = \begin{cases} t, & t \in (0,\infty) \\ 0, & \text{otherwise} \end{cases}$$

. Now we have from (19), we get $(u \circ v)(t) = \Delta_h^{-1} e^{-\frac{1}{\tau^{\nu}}(\xi - t)} \xi \,|_{\xi=0}^{\infty}$.

Then using (17), which gives $(u \circ v)(t) = \dfrac{h^2 e^{-\frac{1}{\tau^{\nu}}(h-t)}}{\tau(e^{-\frac{h}{\tau^{\nu}}} - 1)^2}$.

Then the relation as follows

$$(u \circ v)(t) = \frac{h}{\tau} \sum_{r=0}^{\infty} (rh) e^{-\frac{1}{\tau^{\nu}}(rh-t)} = \frac{h^2 e^{-\frac{1}{\tau^{\nu}}(h-t)}}{\tau(e^{-\frac{h}{\tau^{\nu}}} - 1)^2}$$

which verified for the values $t = 2, h = 3$, $\tau = 2$, and $\nu = 0.4$ by (5) MATLAB coding given bellow: $\begin{array}{l}(9.*exp(-1./2.\wedge(0.4)))./(2.*(exp(-3./2.^{(}0.4))-1).\wedge(2)) = \\ (3./2).*symsum(r.*3.*exp(-(3.*r-2)./(2.\wedge(0.4))), r, 0, inf)\end{array}$.

The following Figure.3 explains the input time(t) function $u(t)$ and $v(t)$ and Figure.4 tells that the convolution product of the functions in the frequency (τ) domain as varying ν as $0.4, 0.5, 0.3$ which are generated by MATLAB are shown below.

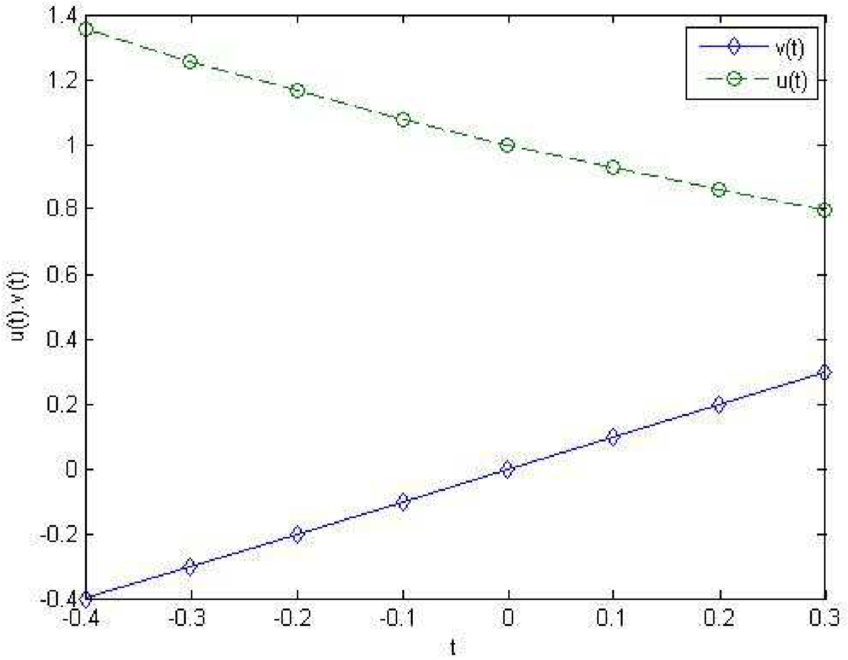

Figure 3: Time(t)

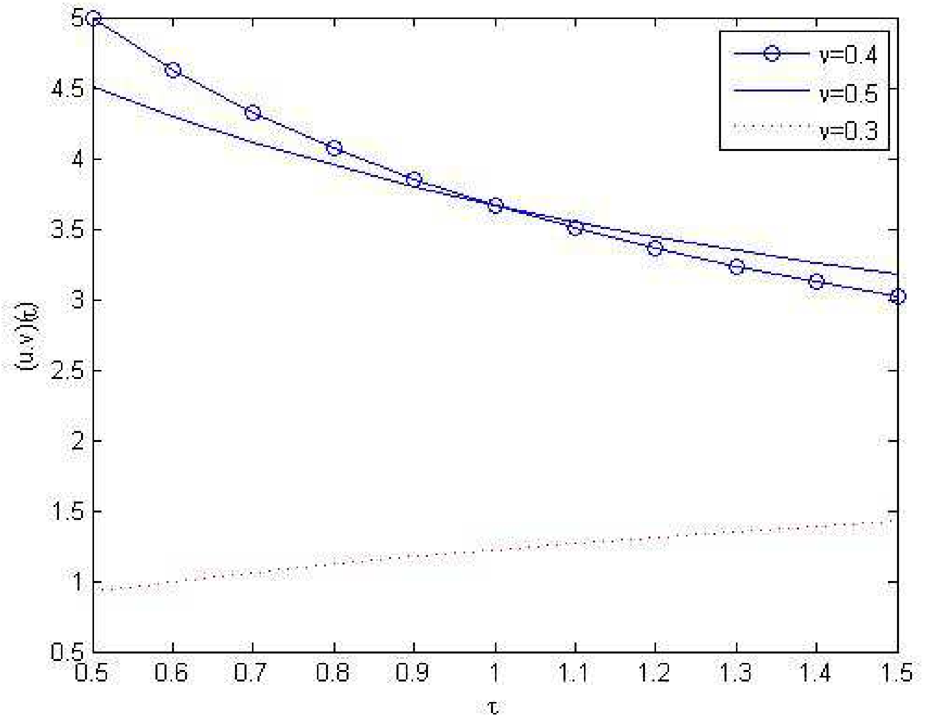

Figure 4: Frequency (τ)

5 Conclusion

In this research, we successfully define a new type of fractional frequency Sumudu transform with shift value $'h'$ several results on fractional frequency Sumudu transform are derived using inverse difference operator and which are verified and analysed by MATLAB. We believe that this transform is an alternative in the field of difference equations. The more advantage of this research is when $\nu = 1$ and $h \to 0$, we get the same results on classical Sumudu transform which is existing in the literature.

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