Solutions of field equations $R_{ij}=0$ for $[\frac{1}{\sqrt{2}}(y+z)-t]$-type and $[t\sqrt{2}/(y+z)]$-type waves in five dimensional space-time (I)

[a]I. S. Mohurley, [b]J. K. Jumale
[a]Head, Department of Physics, Shri Dnyanesh Mahavidyalaya, Nawargaon - 441 223, (M. S.), India
Kaveri Nagar, Yavatmal road,Darwha, Distt.Yavatmal, (M. S.), India.
[a]R.S. Bidkar Arts, Comm. and Science College, Hinganghat, Distt. Wardha, MS, India

Abstract

Investigating five dimensional plane symmetric space-time, plane wave solutions of field equations $R_{ij}=0$ for $[\frac{1}{\sqrt{2}}(y+z)-t]$-type and $[t\sqrt{2}/(y+z)]$-type plane waves are obtained as

$$P=\frac{\bar{\bar{m}}}{m}-\frac{\bar{m}^2}{2m^2}-\frac{\bar{m}\bar{B}}{mB}+\frac{\bar{\bar{B}}}{2B}-\frac{3\bar{B}^2}{4B^2}=0\,.$$

Furthermore, an equivalent plane wave solutions are obtained by using the concept of curvature tensor and Ricci tensor for the same space-time.

§ 1. Introduction

In the paper refer it to [1], Thengane and Ladke (2002) have obtained plane wave solutions g_{ij} of the field equations $R_{ij}=0$ in five dimensional space-time V_5 for general theory of relativity by reformulating Takeno's(1961) definition of plane wave as follows :

Definition : A plane wave g_{ij} is a non-flat solution of the field equations

$$R_{ij}=0,\quad (i,j=1,2,3,4,5) \tag{1.1}$$

in an empty region of the space-time such that

$$g_{ij}=g_{ij}(Z),\qquad Z=Z(x^i),\qquad [x^i=u,x,y,z,t] \tag{1.2}$$

in some suitable co-ordinate system such that

$$g^{ij}Z_{,i}Z_{,j}=0,\qquad Z_{,i}=\frac{\partial Z}{\partial x^i}, \tag{1.3}$$

$$Z=Z(y,z,t),\quad Z_{,3}\neq 0,\quad Z_{,4}\neq 0,\quad Z_{,5}\neq 0\,. \tag{1.4}$$

In this definition, the signature convention adopted is

$$g_{rr} < 0, \qquad r = 1,2,3,4$$

$$\begin{vmatrix} g_{rr} & g_{rs} \\ g_{sr} & g_{ss} \end{vmatrix} > 0, \qquad \begin{vmatrix} g_{rr} & g_{rs} & g_{rt} \\ g_{sr} & g_{ss} & g_{st} \\ g_{tr} & g_{ts} & g_{tt} \end{vmatrix} < 0$$

[not summed for $r, s, t = 1,2,3,4$]

$$\begin{vmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{vmatrix} > 0, \qquad g_{55} > 0 \tag{1.5}$$

and according to $g = \det(g_{ij}) > 0$ (1.6)

The field equation (1.1) then yield

$$\bar{\bar{\omega}}_3\rho_{\alpha\beta} + \bar{\omega}_3\sigma_{\alpha\beta} = 0 = \bar{\bar{\phi}}_3\rho_{\alpha\beta} + \bar{\phi}_3\sigma_{\alpha\beta}, \quad \bar{\bar{\omega}}_4\rho_{\alpha\beta} + \bar{\omega}_4\sigma_{\alpha\beta} = 0 = \bar{\bar{\phi}}_4\rho_{\alpha\beta} + \bar{\phi}_4\sigma_{\alpha\beta} \tag{1.7}$$

where $\phi_3 = \dfrac{Z_{,3}}{Z_{,5}}, \qquad \phi_4 = \dfrac{Z_{,4}}{Z_{,5}},$ (1.8)

$$\omega_3 = t + \phi_3 y, \qquad \omega_4 = t + \phi_4 y, \tag{1.9}$$

$$\sigma_{\alpha\beta} = -\rho_{\alpha\beta} + \frac{1}{4}[\phi_\alpha\phi_\beta L_1 - 2L_2(\phi_\beta\rho_\alpha + \phi_\alpha\rho_\beta) + 2\rho_\alpha\rho_\beta], \tag{1.10}$$

$$\rho_{\alpha\beta} = \frac{1}{2}[\phi_\alpha\rho_\beta + \phi_\beta\rho_\alpha] - \phi_\alpha\phi_\beta L_2. \tag{1.11}$$

Furthermore, the existence of $[\frac{1}{\sqrt{2}}(y+z) - t]$-type and $[t\sqrt{2}/(y+z)]$-type waves in V_5 has been established in the paper refer it to [2] and the solutions (1.7) reduced to

$$\bar{L}_2 - \bar{\rho}_5 + \frac{\rho_5^2}{2} - L_2\rho_5 + \frac{L_1}{4} = 0. \tag{1.12}$$

Considering the proper co-ordinate system in five dimension as

$$g^{ij} = 0, \qquad g_{ij} = 0, \qquad i \neq j, \qquad \text{for } i, j = 1,2,3,4,5 \tag{1.13}$$

we have investigated the general line element in V_5 as

$$ds^2 = \sum_{i=1}^{2}(-A)(dx^i)^2 - 2\phi_3^2 B dy^2 - 2\phi_4^2 B dz^2 + B dt^2$$

i.e. $ds^2 = -Adu^2 - Adx^2 - 2\phi_3^2 Bdy^2 - 2\phi_4^2 Bdz^2 + Bdt^2$. (1.14)

In the present paper, we have studied the plane wave solutions (1.12) in detail for $[\frac{1}{\sqrt{2}}(y+z)-t]$-type and $[t\sqrt{2}/(y+z)]$-type plane waves using the space-time (1.14).

§ 2. $[\frac{1}{\sqrt{2}}(y+z)-t]$ -type plane wave in V_5

For $[\frac{1}{\sqrt{2}}(y+z)-t]$-type plane wave, the line element (1.14) becomes

$$ds^2 = -Adu^2 - Adx^2 - Bdy^2 - Bdz^2 + Bdt^2 . \tag{2.1}$$

where A and B are functions of $Z = [\frac{1}{\sqrt{2}}(y+z)-t]$

$$\phi_3 = \frac{Z_{,3}}{Z_{,5}} = -\frac{1}{\sqrt{2}}, \quad \phi_4 = \frac{Z_{,4}}{Z_{,5}} = -\frac{1}{\sqrt{2}} . \tag{2.2}$$

Then we get

$$\omega^i = \phi_\alpha g^{\alpha i} = [0,0,\frac{1}{B\sqrt{2}},\frac{1}{B\sqrt{2}},\frac{1}{B}], \tag{2.3}$$

$$\rho_i = \bar{g}_{ij}\omega^j = [0,0,\frac{-\bar{B}}{B\sqrt{2}},\frac{-\bar{B}}{B\sqrt{2}},\frac{-\bar{B}}{B}], \tag{2.4}$$

$$L_2 = \frac{\bar{m}}{m} + \frac{3\bar{B}}{2B}, \tag{2.5}$$

$$L_1 = \frac{2\bar{m}^2}{m^2} + \frac{3\bar{B}^2}{B^2} . \tag{2.6}$$

The field equations (1.12) then yield

$$P = \frac{\bar{\bar{m}}}{m} - \frac{\bar{m}^2}{2m^2} - \frac{\bar{m}\bar{B}}{mB} + \frac{\bar{\bar{B}}}{2B} - \frac{3\bar{B}^2}{4B^2} = 0 . \tag{2.7}$$

§ 3. $[t\sqrt{2}/(y+z)]$ -type plane wave in V_5

For $[t\sqrt{2}/(y+z)]$ -type plane wave, the line element (1.14) becomes

$$ds^2 = -Adu^2 - Adx^2 - BZ^2dy^2 - BZ^2dz^2 + Bdt^2 \tag{3.1}$$

where A and B are functions of $Z = [t\sqrt{2}/(y+z)]$ with

$$\phi_3 = \frac{Z_{,3}}{Z_{,5}} = -\frac{Z}{\sqrt{2}}, \qquad \phi_4 = \frac{Z_{,4}}{Z_{,5}} = -\frac{Z}{\sqrt{2}}. \tag{3.2}$$

Then we get

$$\omega^i = \phi_\alpha g^{\alpha i} = [0,0,\frac{1}{ZB\sqrt{2}},\frac{1}{ZB\sqrt{2}},\frac{1}{B}], \tag{3.3}$$

$$\rho_i = \bar{g}_{ij}\omega^j = [0,0,\frac{-1}{\sqrt{2}}(2+\frac{Z\bar{B}}{B}),\frac{-1}{\sqrt{2}}(2+\frac{Z\bar{B}}{B}),\frac{\bar{B}}{B}], \tag{3.4}$$

$$L_2 = \frac{\bar{m}}{m} + \frac{3\bar{B}}{2B} + \frac{2}{Z}, \tag{3.5}$$

$$L_1 = \frac{2\bar{m}^2}{m^2} + \frac{3\bar{B}^2}{B^2} + \frac{8\bar{B}}{ZB} + \frac{8}{Z^2}. \tag{3.6}$$

The field equations (1.12) then yield

$$P = \frac{\bar{\bar{m}}}{m} - \frac{\bar{m}^2}{2m^2} - \frac{\bar{m}\bar{B}}{mB} + \frac{\bar{\bar{B}}}{2B} - \frac{3\bar{B}^2}{4B^2} = 0. \tag{3.7}$$

The results (2.7) and (3.7) for $[\frac{1}{\sqrt{2}}(y+z)-t]$-type and $[t\sqrt{2}/(y+z)]$-type plane waves can also be obtained by employing the concept of curvature tensor and Ricci tensor as under.

§ 4. $[\frac{1}{\sqrt{2}}(y+z)-t]$ -type plane wave in V_5

Non-vanishing components of Christoffel's symbols from (2.1) are calculated as follows :

$$\Gamma^3_{11} = \Gamma^3_{22} = \Gamma^4_{11} = \Gamma^4_{22} = \frac{-\bar{A}}{2B\sqrt{2}}, \qquad \Gamma^5_{11} = \Gamma^5_{22} = \frac{-\bar{A}}{2B},$$

$$\Gamma^1_{13} = \Gamma^2_{23} = \Gamma^1_{14} = \Gamma^2_{24} = \frac{\bar{A}}{2A\sqrt{2}},$$

$$\Gamma^1_{15} = \Gamma^2_{25} = \frac{-\bar{A}}{2A}, \qquad \Gamma^5_{33} = \Gamma^5_{44} = \Gamma^5_{55} = \Gamma^3_{35} = \Gamma^4_{45} = \frac{-\bar{B}}{2B},$$

$$-\Gamma^3_{33} = \Gamma^4_{33} = \Gamma^3_{44} = -\Gamma^4_{44} = -\Gamma^3_{55} = -\Gamma^4_{55} = -\Gamma^3_{34} = -\Gamma^4_{34} = -\Gamma^5_{35} = -\Gamma^5_{45} = \frac{-\bar{B}}{2B\sqrt{2}} . \quad (4.1)$$

Non-vanishing components of curvature tensor in V_5 are as under.

$$2R_{1313} = 2R_{2323} = 2R_{1314} = 2R_{2324} = 2R_{1414} = 2R_{2424} = R_{1515} = R_{2525} = [\frac{\bar{\bar{A}}}{2} - \frac{\bar{A}^2}{4A} - \frac{\bar{A}\bar{B}}{2B}]$$

$$R_{1315} = R_{2325} = R_{1415} = R_{2425} = -\frac{1}{\sqrt{2}}[\frac{\bar{\bar{A}}}{2} - \frac{\bar{A}^2}{4A} - \frac{\bar{A}\bar{B}}{2B}],$$

$$R_{3434} = -\sqrt{2}R_{3435} = \sqrt{2}R_{3445} = 2R_{3535} = -2R_{3545} = 2R_{4545} = [\frac{\bar{\bar{B}}}{2} - \frac{3\bar{B}^2}{4B}]. \quad (4.2)$$

And non-vanishing components of Ricci tensor are calculated as under

$$2R_{33} = 2R_{44} = R_{55} = 2R_{34} = -\sqrt{2}R_{35} = -\sqrt{2}R_{45} = [\frac{\bar{\bar{A}}}{A} - \frac{\bar{A}^2}{2A^2} - \frac{\bar{A}\bar{B}}{AB} + \frac{\bar{\bar{B}}}{B} - \frac{3\bar{B}^2}{4B^2}].$$

These values are related as

$$2R_{33} = 2R_{44} = R_{55} = 2R_{34} = -\sqrt{2}R_{35} = -\sqrt{2}R_{45} = P \quad (4.3)$$

where $P = [\frac{\bar{\bar{A}}}{A} - \frac{\bar{A}^2}{2A^2} - \frac{\bar{A}\bar{B}}{AB} + \frac{\bar{\bar{B}}}{B} - \frac{3\bar{B}^2}{4B^2}].$ (4.4)

The field equation (1.12) becomes

$$\frac{\bar{\bar{A}}}{A} - \frac{\bar{A}^2}{2A^2} - \frac{\bar{A}\bar{B}}{AB} + \frac{\bar{\bar{B}}}{B} - \frac{3\bar{B}^2}{4B^2} = 0 \quad (4.5)$$

which is equivalent to (2.7).

§ 5. $[t\sqrt{2}/(y+z)]$ **-type plane wave in** V_5

Non-vanishing components of Christoffel's symbols from (3.1) are calculated as follows :

$$\Gamma^3_{11} = \Gamma^3_{22} = \Gamma^4_{11} = \Gamma^4_{22} = \frac{1}{y+z}\left(\frac{\bar{A}}{2BZ}\right), \qquad \Gamma^5_{11} = \Gamma^5_{22} = \frac{1}{y+z}\left(\frac{\bar{A}}{B\sqrt{2}}\right),$$

$$\Gamma^1_{13} = \Gamma^2_{23} = \Gamma^1_{14} = \Gamma^2_{24} = \frac{1}{y+z}\left(-\frac{Z\bar{A}}{2A}\right), \qquad \Gamma^1_{15} = \Gamma^2_{25} = \frac{1}{y+z}\left(\frac{\bar{A}}{A\sqrt{2}}\right),$$

$$\Gamma^3_{33} = -\Gamma^4_{33} = -\Gamma^3_{44} = \Gamma^4_{44} = \Gamma^3_{34} = \Gamma^4_{34} = -\frac{1}{y+z}\left(1+\frac{Z\overline{B}}{2B}\right), \quad \Gamma^5_{33} = \frac{Z\sqrt{2}}{y+z}\left(1+\frac{Z\overline{B}}{2B}\right),$$
$$\Gamma^3_{55} = \Gamma^4_{55} = \frac{1}{y+z}\left(\frac{-\overline{B}}{2ZB}\right), \quad \Gamma^5_{55} = \frac{1}{y+z}\left(\frac{\overline{B}}{B\sqrt{2}}\right), \quad \Gamma^5_{44} = \frac{Z\sqrt{2}}{y+z}\left(1+\frac{Z\overline{B}}{2B}\right),$$
$$\Gamma^3_{35} = \Gamma^4_{45} = \frac{\sqrt{2}}{y+z}\left(\frac{1}{Z}+\frac{\overline{B}}{2B}\right), \quad \Gamma^3_{35} = \Gamma^5_{45} = \frac{1}{y+z}\left(-\frac{Z\overline{B}}{2B}\right). \qquad (5.1)$$

Non-vanishing components of curvature tensor in V_5 are as under.

$$R_{1313} = R_{2323} = R_{1314} = R_{2324} = \frac{Z^2}{(y+z)^2}[\frac{\overline{\overline{A}}}{2} - \frac{\overline{A}^2}{4A} - \frac{\overline{A}\overline{B}}{2B}],$$

$$R_{1315} = R_{2325} = R_{1415} = R_{2425} = \frac{-Z\sqrt{2}}{(y+z)^2}[\frac{\overline{\overline{A}}}{2} - \frac{\overline{A}^2}{4A} - \frac{\overline{A}\overline{B}}{2B}],$$

$$R_{1414} = R_{2424} = \frac{Z^2}{(y+z)^2}[\frac{\overline{\overline{A}}}{2} - \frac{\overline{A}^2}{4A} - \frac{\overline{A}\overline{B}}{2B}],$$

$$R_{1515} = R_{2525} = \frac{2}{(y+z)^2}[\frac{\overline{\overline{A}}}{2} - \frac{\overline{A}^2}{4A} - \frac{\overline{A}\overline{B}}{2B}],$$

$$-R_{3435} = R_{3445} = \frac{Z^3\sqrt{2}}{(y+z)^2}[\frac{\overline{\overline{B}}}{2} - \frac{3\overline{B}^2}{4B}],$$

$$R_{3535} = -R_{3545} = R_{4545} = \frac{Z^2}{(y+z)^2}[\frac{\overline{\overline{B}}}{2} - \frac{3\overline{B}^2}{4B}]. \qquad (5.2)$$

And non-vanishing components of Ricci tensor are calculated as under

$$R_{33}/Z^2 = R_{44}/Z^2 = R_{55}/2 = R_{34}/Z^2 = -R_{35}/Z\sqrt{2} = -R_{45}/Z\sqrt{2} = \frac{1}{(y+z)^2}P.$$

These values are related as

$$\frac{(y+z)^2}{Z^2}R_{33} = \frac{(y+z)^2}{Z^2}R_{44} = \frac{(y+z)^2}{2}R_{55} = \frac{(y+z)^2}{Z^2}R_{34} = -\frac{(y+z)^2}{Z\sqrt{2}}R_{35}$$
$$= -\frac{(y+z)^2}{Z\sqrt{2}}R_{45} = P \qquad (5.3)$$

where $P = [\frac{\bar{\bar{A}}}{A} - \frac{\bar{A}^2}{2A^2} - \frac{\bar{A}\bar{B}}{AB} + \frac{\bar{\bar{B}}}{B} - \frac{3\bar{B}^2}{4B^2}]$. (5.4)

The field equation (1.12) becomes

$$\frac{\bar{\bar{A}}}{A} - \frac{\bar{A}^2}{2A^2} - \frac{\bar{A}\bar{B}}{AB} + \frac{\bar{\bar{B}}}{B} - \frac{3\bar{B}^2}{4B^2} = 0 \quad (5.5)$$

which is equivalent to (3.7).

Conclusion : *The plane wave solutions of field equations* $R_{ij} = 0$ *in five dimensional space-time* V_5 *for* $[\frac{1}{\sqrt{2}}(y+z)-t]$*-type and* $[\frac{t}{\sqrt{2}}/(y+z)]$*-type plane waves can be obtained by using the concept of curvature tensor as well as without using the concept of curvature tensor and found that both the results are equivalent to each other.*

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