$[z-\frac{1}{\sqrt{2}}(t_1+t_2)]$-Type and $[(t_1+t_2)/z\sqrt{2}]$ -Type Exact Plane Wave Solutions of Einstein Maxwell Field Equations in Four Dimensional Space-Time

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Abstract

The most general solutions of Einstein Maxwell field equations in four-dimensional space-times V_4 having two time axes in general theory of relativity are given by (g_{ij}, F_{ij}) satisfying

$P = 8\pi\sigma^2/m$ and $P = -8\pi\sigma^2/mZ^2$

Corresponding to $[z-\frac{1}{\sqrt{2}}(t_1+t_2)]$-type and $[(t_1+t_2)/z\sqrt{2}]$ -type plane waves respectively

where m and σ are functions of Z.

§1. Introduction

Takeno (1961) has solved field equations $R_{ij} = 0$ and obtained solutions g_{ij} in the form

$$P = \overline{\overline{m}}/2m - \overline{m}^2/4m^2 - \overline{m}\,\overline{C}/2mC - (\overline{A}\,\overline{B} - \overline{D}^2)/2m = 0 \quad \text{and} \tag{1.1}$$

$$P' = \overline{\overline{m}}/2m - \overline{m}^2/4m^2 - \overline{m}\,\overline{C}/2mC - (\overline{A}\,\overline{B} - \overline{D}^2)/2m + E\overline{m}/2mCZ = 0 \tag{1.2}$$

for $(z-t)$-type and (t/z)-type plane waves respectively in an empty region of four dimensional space-time in V_4. Furthermore he has investigated the general solutions (g_{ij}, F_{ij}) of the Einstein Maxwell field equations

$$R_{ij} = -8\pi E_{ij}, \quad i, j = 1,2,3,4 \tag{1.3}$$

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \tag{1.4}$$

$$F^{ij}_{;j} = 0 \text{ which satisfy} \tag{1.5}$$

$$P = -\frac{8\pi}{m}[A\rho^2 + 2D\rho\sigma + B\sigma^2] \text{ and } P' = -\frac{8\pi}{Z^2 m}[A\rho^2 + 2D\rho\sigma + B\sigma^2] \qquad (1.6)$$

corresponding to $(z-t)$-type and (t/z)- type waves respectively. Here P and P' are defined as (1.1) and (1.2) respectively and $Z = Z(z-t)$ and $Z = (t/z)$, ρ and σ are the functions of Z and (;) denotes the covariant derivative. In the paper refer it to [1], we have obtained plane wave solutions of the field equations $R_{ij} = 0$ in an empty region for $[z - \frac{1}{\sqrt{2}}(t_1 + t_2)]$-type and $[(t_1 + t_2)/z\sqrt{2}]$ -type plane waves in V_4 having two time axes on the lines of Takeno(1961) in the form

$$P = \frac{\overline{\overline{m}}}{2m} - \frac{\overline{m}^2}{4m^2} - \frac{\overline{m}\,\overline{B}}{2mB} + \frac{\overline{\overline{B}}}{2B} - \frac{3\overline{B}^2}{4B^2} = 0 \qquad (1.7)$$

where m and B are functions of Z .

It is to be noted that this plane wave solutions (1.7) for these two types of plane waves can further be generalized by considering the case where an electromagnetic field coexists with the gravitational field and therefore we have obtained the most general exact plane wave solutions of Einstein Maxwell field equations for these two types of plane waves in four dimensional space-times V_4 having two time axes in the present paper.

2. $[z - \frac{1}{\sqrt{2}}(t_1 + t_2)]$-type plane wave solutions

For $[z - \frac{1}{\sqrt{2}}(t_1 + t_2)]$-type plane wave, we consider the metric

$$ds^2 = -Ady^2 - 2Bdz^2 + 2Bdt_1^2 + 2Bdt_2^2 \qquad (2.1)$$

where A, B are functions of $Z = [z - \frac{1}{\sqrt{2}}(t_1 + t_2)]$.

Here we study the space-time (2.1) with the condition that an electromagnetic field co-exists with the gravitational field such that wave propogation is along the positive direction of z-axis and then obtain the solutions of the field equations (1.3), (1.4) and (1.5) corresponding to two different cases such that

$$F_{ij} = F_{ij}(Z) \quad \text{or} \quad K_i = K_i(Z)$$

In the co-ordinate system under consideration.

Case I : F_{ij} is transverse electromagnetic and its components are functions of Z alone. Since E_{ij} is transverse electromagnetic, the electric and magnetic field components of F_{ij} along z-direction vanish.

$$E_z = H_z = 0 \quad \text{or} \quad F_{23} = F_{24} = 0. \tag{2.2}$$

From (1.4) and (2.2), we obtain

$$F_{12} = \sigma, \qquad F_{13} = -\frac{1}{\sqrt{2}}(\sigma + C_1), \qquad F_{14} = -\frac{1}{\sqrt{2}}(\sigma + C_2),$$

$$F_{34} = C_3, \qquad F_{23} = 0, \qquad F_{24} = 0 \tag{2.3}$$

and C_1, C_2, C_3 are arbitrary constants and σ is arbitrary function of Z.

Using (2.1), (2.2) and (2.3), we get

$$F^{12} = -\frac{\sigma}{2Bm}, \qquad F^{13} = -\frac{1}{2Bm\sqrt{2}}(\sigma + C_1), \qquad F^{14} = -\frac{1}{2Bm\sqrt{2}}(\sigma + C_2)$$

$$F^{34} = -\frac{AC_3}{4B^2 m}. \tag{2.4}$$

Equation (2.3) and (2.4) give

$$F_{ij}F^{ij} = \frac{1}{2mB^2}[C_1 B(2\sigma + C_1) + C_2 B(2\sigma + C_2) - AC_3^2]. \tag{2.5}$$

The nonzero components of the electromagnetic energy tensor

$$E_{ij} = \frac{1}{4} g_{ij} F_{kl} F^{kl} - F_{ik} F_{jl} g^{kl} \tag{2.6}$$

are $$E_{11} = \frac{1}{8mB^2}[ABC_1(2\sigma + C_1) + ABC_2(2\sigma + C_2) + A^2 C_3^2],$$

$$E_{22} = \frac{1}{4mB^2}[-B^2 C_1(2\sigma + C_1) - B^2 C_2(2\sigma + C_2) + ABC_3^2 - 4B^2\sigma^2],$$

$$E_{33} = \frac{1}{4mB^2}[-B^2 C_1(2\sigma + C_1) + B^2 C_2(2\sigma + C_2) + ABC_3^2 - 2B^2\sigma^2],$$

$$E_{44} = \frac{1}{4mB^2}[B^2 C_1(2\sigma + C_1) - B^2 C_2(2\sigma + C_2) + ABC_3^2 - 2B^2\sigma^2],$$

$$E_{23} = \frac{1}{mB^2\sqrt{2}}[B^2\sigma(\sigma + C_1)], \qquad E_{24} = \frac{1}{mB^2\sqrt{2}}[B^2\sigma(\sigma + C_2)],$$

$$E_{34} = \frac{1}{2mB^2}[-B^2\sigma(\sigma + C_1 + C_2) - B^2 C_1 C_2]. \tag{2.7}$$

The relation of the nonzero components of Ricci tensor obtained in [1] are as under

$$R_{22} = 2R_{33} = 2R_{44} = -\sqrt{2}R_{23} = -\sqrt{2}R_{24} = 2R_{34} \ . \tag{2.8}$$

From (1.3) and (2.8), we get

$$E_{22} = 2E_{33} = 2E_{44} = -\sqrt{2}E_{23} = -\sqrt{2}E_{24} = 2E_{34} \tag{2.9}$$

which further yield

$$2\sigma C_1 B - 6\sigma C_2 B + BC_1^2 - 3BC_2^2 - AC_3^2 = 0 \tag{2.10}$$

Confining to the real quantities, we have $C_1 = C_2 = C_3 = 0$. (2.11)

Using (2.10), the equations (2.3) and (2.4) gives

$$F_{12} = \sigma, \quad F_{13} = -\frac{\sigma}{\sqrt{2}}, \qquad F_{14} = -\frac{\sigma}{\sqrt{2}}, \qquad F_{34} = 0. \tag{2.12}$$

and

$$F^{12} = -\frac{\sigma}{2Bm}, \qquad F^{13} = -\frac{\sigma}{2Bm\sqrt{2}}, \qquad F^{14} = -\frac{\sigma}{2Bm\sqrt{2}}, \qquad F^{34} = 0. \tag{2.13}$$

Therefore equations (2.7), yield

$$E_{22} = 2E_{33} = 2E_{44} = -\sqrt{2}E_{23} = -\sqrt{2}E_{24} = 2E_{34} = -\sigma^2/m \tag{2.14}$$

and $E_{1\alpha} = 0$, $\alpha = 2, 3, 4$. (2.15)

And (1.3) is reduced to the only one equation

$$P = 8\pi\sigma^2/m. \tag{2.16}$$

Conversely, F_{ij} given by (2.12) satisfies (1.4) identically and moreover g_{ij} given by (2.1) and this F_{ij} satisfy the equation (1.3) if and only if they satisfy the equations (2.16) where P is defined by (1.7).

Now we shall consider the field equation (1.5), we have

$$\left\{ \begin{matrix} j \\ 1j \end{matrix} \right\} = 0, \left\{ \begin{matrix} j \\ 2j \end{matrix} \right\} = -\sqrt{2}\left\{ \begin{matrix} j \\ 3j \end{matrix} \right\} = -\sqrt{2}\left\{ \begin{matrix} j \\ 4j \end{matrix} \right\} = -\frac{1}{M\sqrt{2}}[\frac{\overline{m}}{m} + \frac{3\overline{B}}{B}]. \qquad (2.17)$$

Then we have from (2.13) and (2.17)

$$F^{ij}_{;j} = 0. \qquad (2.18)$$

Therefore field equation (1.5) is satisfied identically.

Case II The four potentials K_i are the functions of $Z = [z - \frac{1}{\sqrt{2}}(t_1 + t_2)]$**.**

Noting $F_{ij} = K_{j,i} - K_{i,j}$, we obtain

$$F_{12} = \sigma, \qquad F_{13} = -\frac{\sigma}{\sqrt{2}}, \qquad F_{14} = -\frac{\sigma}{\sqrt{2}}, \qquad F_{34} = 0.$$

$$F_{23} = \frac{\rho}{\sqrt{2}} + \eta, \qquad F_{24} = \frac{\rho}{\sqrt{2}} + \tau, \qquad F_{34} = \frac{1}{\sqrt{2}}(-\tau + \eta) \qquad (2.19)$$

and $\sigma = -\overline{K}_1, \quad \rho = \overline{K}_2, \quad \eta = \overline{K}_3, \quad \tau = \overline{K}_4$

and σ, η, τ and ρ are the functions of Z. (2.20)

Also $F^{12} = -\frac{\sigma}{2Bm}, \quad F^{13} = -\frac{\sigma}{2Bm\sqrt{2}}, \quad F^{14} = -\frac{\sigma}{2Bm\sqrt{2}},$

$$F^{23} = -\frac{1}{4B^2}(\eta + \frac{\rho}{\sqrt{2}}), \quad F^{24} = -\frac{1}{4B^2}(\tau + \frac{\rho}{\sqrt{2}}), \quad F^{34} = \frac{1}{4B^2}(-\frac{\tau}{\sqrt{2}} + \frac{\eta}{\sqrt{2}}). \qquad (2.21)$$

Equation (2.19) and (2.21) give

$$F_{ij}F^{ij} = -\frac{1}{4B^2}[\eta(\eta + 2\rho\sqrt{2}) + 2\rho(\rho + \tau\sqrt{2}) + \tau(\tau + 2\eta)] \qquad (2.22)$$

and $\frac{1}{\sqrt{2}}E_{12} = -E_{13} = -E_{14} = \frac{\sigma}{4B}[\eta + \sqrt{2}\rho + \tau]$ (2.23)

But, $E_{1\alpha}=0$ follows from (1.3), (2.1) and (2.8). Therefore we have the following two conditions either $\rho=\eta=\tau=0$ or $\rho\neq 0,\eta\neq 0,\tau\neq 0$.

Condition 1 If we assume $\rho=\eta=\tau=0$ then F_{ij} become (2.12).

Condition 2 If we assume $\rho\neq 0,\eta\neq 0,\tau\neq 0$ then in this case we must have $\sigma=0$.

Therefore $$E_{11}=(-g_{11})\frac{1}{16B^2}[\eta(\eta+2\rho\sqrt{2})+2\rho(\rho+\tau\sqrt{2})+\tau(\tau+2\eta)] \quad (2.24)$$

which is contradiction to the fact that $E_{11}=0$ which is obtained from (1.3), (2.1) and (2.8). Hence we can not have this type of condition.

Conclusion *The most general solution of the Einstein Maxwell field equations* (1.3), (1.4) *and* (1.5) *in four dimensional space-times* V_4 *having two time axes are given by* (2.1), (2.12) *under only one constraint* (2.16) *such that either Maxwell tensor* F_{ij} *is a transverse electromagnetic and its components are function of Z alone or the components of the four potential* K_i *are functions of* $Z=[z-\frac{1}{\sqrt{2}}(t_1+t_2)]$ *in the coordinate system under consideration.*

It is interesting to note that if $F_{ij}=0$ this result is reduced to (1.7), which is obtained in the case of purely gravitational field.

Geometric property of the solution (g_{ij},F_{ij})

[I] Let (g_{ij},F_{ij}) be a solution of the Einstein Maxwell field equations obtained in earlier section then electromagnetic field F_{ij} is null.

Proof Let F^*_{ij} is the dual tensor of F_{ij} defined by $F^*_{ij}=\eta_{ijkl}F^{kl}/2$,

η_{ijkl} being the tensor which is antisymmetric with respect to each pair of indices and $\eta_{1234}=\sqrt{-g}$.

Then from (2.1) and (2.12) we have

$$F^*_{12}=0,\ F^*_{13}=0,\ F^*_{14}=0,\quad F^*_{23}=-\frac{\sigma\sqrt{B}}{\sqrt{m}},\ F^*_{24}=-\frac{\sigma\sqrt{B}}{\sqrt{m}},\ F^*_{34}=-\frac{\sigma\sqrt{2}\sqrt{B}}{\sqrt{m}} \quad (2.25)$$

and $$F_{ij}F^{ij}=0,\ F^*_{ij}F^{ij}=0. \quad (2.26)$$

Equation (2.26) shows that the electromagnetic field F_{ij} is null.

3. $[(t_1+t_2)/z\sqrt{2}]$ -type plane wave solutions

For $[(t_1+t_2)/z\sqrt{2}]$ -type plane wave, we consider the metric

$$ds^2 = -Ady^2 - 2Z^2Bdz^2 + 2Bdt_1^2 + 2Bdt_2^2 \tag{3.1}$$

where *A*, *B* are functions of $Z = [(t_1+t_2)/z\sqrt{2}]$.

In this section, we shall show that we can make similar considerations with respective (3.1). Corresponding to case I we assume as follows :

Case III F_{ij} is transverse electromagnetic and its components in the coordinate system under consideration are of the form

$$zF_{ij} = f_{ij}(Z), \quad Z = [(t_1+t_2)/z\sqrt{2}] \tag{3.2}$$

Since F_{ij} is transverse electromagnetic, the electric and magnetic field components of F_{ij} along z-direction vanish.

i.e. $E_z = H_z = 0$ or $F_{23} = F_{24} = 0$. (3.3)

From (1.4) and (3.3), we obtain the first half of the generalized Maxwell equations

$$F_{12} = \sigma, \quad F_{13} = -\left(\frac{z}{t_1+t_2}\right)(\sigma + C_1), \quad F_{14} = -\left(\frac{z}{t_1+t_2}\right)(\sigma + C_1), \quad F_{34} = C_3. \tag{3.4}$$

where C_1, C_2, C_3 are arbitrary constants and σ is arbitrary function of *Z*.

Corresponding to (2.4), (2.5) and (2.7) in the section (2), we have obtained respectively

$$F^{12} = -\frac{\sigma}{2Z^2Bm}, \quad F^{13} = \left(\frac{-z}{t_1+t_2}\right)\left(\frac{\sigma+C_1}{2Bm}\right), \quad F^{14} = \left(\frac{-z}{t_1+t_2}\right)\left(\frac{\sigma+C_1}{2Bm}\right),$$
$$F^{34} = -\frac{AC_3}{4mB^2}, \tag{3.5}$$

$$F_{ij}F^{ij} = \frac{1}{2mZ^2B^2}[C_1B(2\sigma + C_1) + C_2B(2\sigma + C_2) - Z^2AC_3^2] \tag{3.6}$$

and $$E_{11}=\frac{1}{8mB^2Z^2}[ABC_1(2\sigma+C_1)+ABC_2(2\sigma+C_2)+Z^2A^2C_3^2],$$

$$E_{22}=\frac{1}{4mB^2}[-B^2C_1(2\sigma+C_1)-B^2C_2(2\sigma+C_2)+ABC_3^2Z^2-4B^2\sigma^2],$$

$$E_{33}=\frac{1}{4mB^2Z^2}[-B^2C_1(2\sigma+C_1)+B^2C_2(2\sigma+C_2)+ABC_3^2Z^2-2B^2\sigma^2],$$

$$E_{44}=\frac{1}{4mB^2Z^2}[B^2C_1(2\sigma+C_1)-B^2C_2(2\sigma+C_2)+ABC_3^2Z^2-2B^2\sigma^2],$$

$$E_{23}=\frac{1}{mB^2Z\sqrt{2}}[B^2\sigma(\sigma+C_1)],\qquad E_{24}=\frac{1}{mB^2Z\sqrt{2}}[B^2\sigma(\sigma+C_2)],$$

$$E_{34}=\frac{1}{2mB^2Z^2}[-B^2\sigma(\sigma+C_1+C_2)-B^2C_1C_2]. \tag{3.7}$$

The nonzero components of Ricci tensor obtained in [1] are as under

$$R_{22}=2Z^2R_{33}=2Z^2R_{44}=-Z\sqrt{2}R_{23}=-Z\sqrt{2}R_{24}=2Z^2R_{34}. \tag{3.8}$$

From (1.3) and (3.7), we get

$$E_{22}=2Z^2E_{33}=2Z^2E_{44}=-Z\sqrt{2}E_{23}=-Z\sqrt{2}E_{24}=2Z^2E_{34} \tag{3.9}$$

which further yield

$$2\sigma C_1B-6\sigma C_2B+BC_1^2-3BC_2^2-AC_3^2Z^2=0. \tag{3.10}$$

Confining to the real quantities, we have again (2.11).
Then the non-vanishing independent components of F_{ij}, F^{ij} and E_{ij} become

$$F_{12}=\sigma,\quad F_{13}=-(\frac{z}{t_1+t_2})\sigma,\quad F_{14}=-(\frac{z}{t_1+t_2})\sigma,\quad F_{34}=0. \tag{3.11}$$

$$F^{12}=-\frac{\sigma}{2BmZ^2},\quad F^{13}=-(\frac{z}{t_1+t_2})(\frac{\sigma}{2Bm}),\quad F^{14}=-(\frac{z}{t_1+t_2})(\frac{\sigma}{2Bm}),\quad F^{34}=0. \tag{3.12}$$

and $$E_{22}=2Z^2E_{33}=2Z^2E_{44}=-Z\sqrt{2}E_{23}=-Z\sqrt{2}E_{24}=2Z^2E_{34}=-\sigma^2/mZ^2 \tag{3.13}$$

and $E_{1\alpha} = 0$, $\alpha = 2, 3, 4$ (3.14)

where σ is arbitrary function of $Z = [(t_1 + t_2) / z\sqrt{2}]$.

And the field equation (1.3) is equivalent to the only one equation

$$P' = -8\pi\sigma^2 / Z^2 m. \quad (3.15)$$

which corresponds to equation (2.16) for $[z - \frac{1}{\sqrt{2}}[(t_1 + t_2)]$-type wave.

Again (1.4) is satisfied identically by any F_{ij} of the form (3.11).
Now we shall consider the field equation (1.5), we have

$$\begin{Bmatrix} j \\ 1j \end{Bmatrix} = 0, \ \begin{Bmatrix} j \\ 2j \end{Bmatrix} = -Z\sqrt{2}\begin{Bmatrix} j \\ 3j \end{Bmatrix} = -\sqrt{2}\begin{Bmatrix} j \\ 4j \end{Bmatrix} = -\frac{Z}{M\sqrt{2}}[\frac{\overline{m}}{m} + \frac{3\overline{B}}{B} + \frac{2}{Z}]. \quad (3.16)$$

Then we can easily show that the equation (1.5) is satisfied identically by (3.1) and (3.11)

Conclusion Let g_{ij} be given by (3.1). If we assume (3.3), then necessary and sufficient condition that these g_{ij} and F_{ij} satisfy the field equations (1.3) and (1.4) is that F_{ij} be of the form (3.11) and that three functions of Z i.e. A, B, σ satisfy the one condition (3.15). The equation (1.5) is satisfied automatically.

Case IV The components of the four potentials K_i, whose existence is assured by (1.4) are the functions of $Z = [(t_1 + t_2) / z\sqrt{2}]$ in the coordinate system under consideration.
From (2.18), we have (3.2) with

$$-zF_{12} = -f_{12} = \sigma, \qquad zZ\sqrt{2}F_{13} = Z\sqrt{2}f_{13} = \sigma, \qquad zZ\sqrt{2}F_{14} = Z\sqrt{2}f_{14} = \sigma,$$

$$zF_{23} = f_{23} = -(\eta + \frac{\rho}{\sqrt{2}}), \quad zF_{24} = f_{24} = -(\tau + \frac{\rho}{\sqrt{2}}), \quad z\sqrt{2}ZF_{34} = Zf_{34} = \tau - \eta \quad (3.17)$$

where $\sigma = -\overline{K}_1 Z$, $\rho = \overline{K}_2$, $\eta = \overline{K}_3 Z$, $\tau = \overline{K}_4 Z$. (3.18)

and σ, η, τ and ρ are the functions of Z.

Also the expression corresponding to (2.21), (2.22) and (2.23) are under

$$zZF^{12} = \frac{\sigma}{2ZBm}, \qquad zF^{13} = \frac{\sigma}{2BmZ\sqrt{2}}, \qquad zF^{14} = \frac{\sigma}{2BmZ\sqrt{2}},$$

$$zF^{23} = \frac{1}{4Z^2B^2}(\eta + \frac{\rho}{\sqrt{2}}),\quad zF^{24} = -\frac{1}{4Z^2B^2}(\tau + \frac{\rho}{\sqrt{2}}),\quad zF^{34} = \frac{1}{4ZB^2\sqrt{2}}(\tau - \eta). \quad (3.19)$$

$$z^2F_{ij}F^{ij} = -\frac{1}{4Z^2B^2}[\eta(\eta + 2\rho\sqrt{2}) + 2\rho(\rho + \tau\sqrt{2}) + \tau(\tau + 2\eta\sqrt{2})] \quad (3.20)$$

and

$$E_{12} = -Z^2\sqrt{2}E_{13} = -Z^2\sqrt{2}E_{14} = \frac{\sigma}{2Bz^2\sqrt{2}}[\eta + \sqrt{2}\rho + \tau] \quad (3.21)$$

But, $E_{1\alpha} = 0$ follows from (1.3), (3.1) and (3.8). Therefore we have the following two considerations

(i) when $\rho = \eta = \tau = 0$. In this case F_{ij} become (3.11) and (ii) when $\rho \neq 0, \eta \neq 0, \tau \neq 0$. Then we must have $\sigma = 0$. Therefore

$$E_{11} = (-g_{11})\frac{1}{16B^2z^2Z^2}[\eta(\eta + 2\sqrt{2}) + \rho(2\rho + 2\tau\sqrt{2}) + \tau(\tau + 2\eta)] \quad (3.22)$$

which contradicts to the fact $E_{11} = 0$ which is obtained from (1.3), (3.1) and (3.8). Hence we can not consider this case.

Conclusion *The most general solution of the Einstein Maxwell field equations* (1.3), (1.4) *and* (1.5) *in four dimensional space-times* V_4 *having two time axes are given by* (3.1), (3.11) *under only one constraint* (3.15) *such that either Maxwell tensor* F_{ij} *is a transverse electromagnetic and its components are of the form* (3.2) *or the components of the four potential* K_i *are functions of* $Z = [(t_1 + t_2)/z\sqrt{2}]$ *in the coordinate system under consideration.*

Acknowledgement The authors are thankful to Professor K. D. Thengane of India for his constant inspiration.

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