The Class of Triangular Trees is δ-Optimal sw-sum graph but the Class of Ladders is not

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Abstract

In this paper we have proved that the class of triangular tree TT(n+1) is sw-sum graph with sw -sum number $w(TT(n+1)) = 1$. We have also answered the open problem "Does there exist a graph which is an sw-sum graph but not δ-optimal sw-sum graph". We have proved that $W(L_n) = 2\ for\ n \geq 7$.

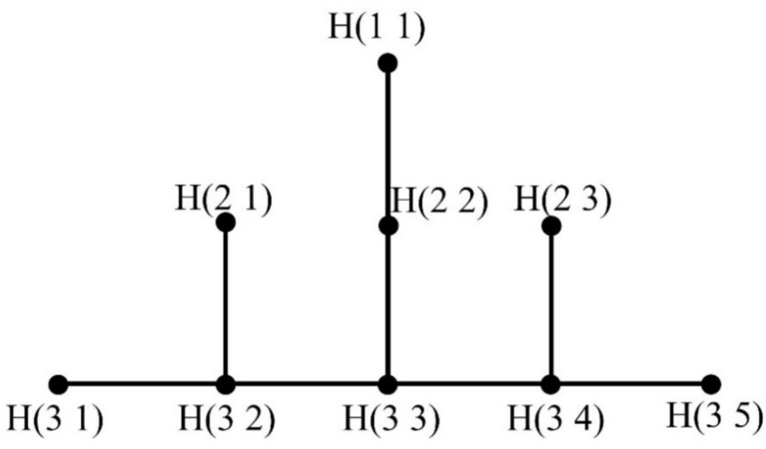
KEYWORDS: sw sum, δ-optimal sw-sum graph, sw sum number, triangular trees, ladders

1. Introduction

In this paper we consider only simple undirected graphs with V(G) to be the set of vertices of graph G & E(G) to be the set of edges of graph G. A super weak sum(sw-sum) labelling is a bijection L: V(G) →{ 1,2,..,|V(G)| } such that for every edge $e = uv$ in E(G) there exist a vertex w in V(G) with L(u) + L(v) = L(w). A graph that can be sw-sum labelled is called sw-sum graph. The sw-sum number w(G) of a connected graph G is the least number r of isolated vertices $\overline{k_r}$ such that H= G U $\overline{k_r}$ is an sw-sum graph. If $w(G) = \delta$ is achieved then G is called δ-optimal sw- sum graph. Imran Javaid, Fariha Khalid, Ali Ahmad & M.Imran[5] showed that not all graphs are sw-sum graphs and they conjectured that all graphs are δ-optimal weak sum graphs. They also posted an open problem "Does there exist a sum graph which is sw-sum graph but not δ-optimal sw-sum graph. In this paper we give the answer to an open problem and also have constructed $W(L_n) = 2$ for $n \geq 7$.

2. Triangular Trees

We call following tree as triangular tree. The vertices are also named in the diagram. The graph and naming style of vertices can be similarly extended for $TT(n+1)$ for all natural numbers n.[1, 4]



TT(3)

Now onwards for an edge $e = uv$, we mean $S(e) = L(u) + L(v)$

Theorem 1: For any odd n, $TT(n+1)$ is 1- optimal sw–sum graph.

Label the vertices of $TT(n+1)$ as follows.
For odd values of t and $1 \le t \le n$
Define

$$L(H(t,t)) = (n+1)^2 - \left(\frac{t-1}{2}\right)(2n - t + 4)$$

$$L(H(t+z,t)) = (n+1)^2 - \left(\frac{t-1}{2}\right)(2n - t + 4) - 2(z-1) - 1$$

$$L(H(t+z, 2(t+z)-t)) = (n+1)^2 - \left(\frac{t-1}{2}\right)(2n - t + 4) - 2(z-1) - 2$$

Where $z = 1$ to $(n+1) - t$
For even values of t and $2 \le t \le (n+1)$

$$L(H(t,t)) = \left(\frac{t}{2} - 1\right)(2n - t + 3) + 1$$

$$L(H(t+z,t)) = \left(\frac{t}{2} - 1\right)(2n - t + 3) + 1 + 2(z-1) + 1$$

$$L(H(t+z, 2(t+z)-t)) = \left(\frac{t}{2} - 1\right)(2n - t + 3) + 1 + 2(z-1) + 2$$

Where $z = 1$ to $(n+1) - t$
Partition the edges of $TT(n+1)$ as follows.
$E_t = \bigcup_{x=t}^{n}\{H(x,t)H(x+1,t+1), H(x,2x-t)H(x+1,2(x+1)-(t+1)\}$,
$B_t = \{H(n+1,t)H(n+1,t+1), H(n+1, 2(n+1-t)H(n+1, 2(n+1)-(t+1))\}$
Hence $E(TT(n+1)) = \bigcup_{t=1}^{n}(E_t \cup B_t)$
Let $e \in E(TT(n+1))$ therefore either $e \in E_t$ or $e \in B_t$ for some natural number t such that $1 \le t \le n$

Case 1) If $e \in E_t$, then for some $z \in \{1,2,\dots,n-t\}$ we have

$$e = H(t,t)H(t+1,t+1) \qquad \ldots(1)$$

Or

$$e = H(t+z,t)H(t+1+z,t+1) \qquad \ldots(2)$$

Or

$$e = H(t+z,2(t+z)-t)H(t+1+z,2(t+1+z)-(t+1)) \qquad \ldots(3)$$

If t is odd and e as in (1) above then

$$S(e) = [(n+1)^2 - \left(\frac{t-1}{2}\right)(2n-t+4)] + [\left(\frac{t+1}{2}-1\right)(2n-(t+1)+3)+1]$$

$$= (n+1)^2 - t + 2 \leq (n+1)^2+1$$

If t is odd and e as in (2) above then

$$S(e) = \left[(n+1)^2 - \left(\frac{t-1}{2}\right)(2n-t+4) - 2(z-1) - 1\right]$$
$$+ [\left(\frac{t+1}{2}-1\right)(2n-(t+1)+3) + 1 + 2(z-1) + 1]$$

$$= (n+1)^2 - t + 2 \leq (n+1)^2+1$$

If t is odd and e as in (3) above then

$$S(e) = \left[(n+1)^2 - \left(\frac{t-1}{2}\right)(2n-t+4) - 2(z-1) - 2\right] +$$
$$[\left(\frac{t+1}{2}-1\right)(2n-(t+1)+3) + 1 + 2(z-1) + 2]$$

$$= (n+1)^2 - t + 2 \leq (n+1)^2+1$$

If t is even and e as in (1) above then

$$S(e) = \left[\left(\frac{t}{2}-1\right)(2n-t+3)+1\right] + \left[(n+1)^2 - \left(\frac{t+1-1}{2}\right)(2n-(t+1)+4)\right]$$

$$= n^2 + t - 1 < n^2 + 2n + 1 = (n+1)^2$$

If t is even and e as in (2) above then

$$S(e) = \left[\left(\frac{t}{2}-1\right)(2n-t+3) + 1 + 2(z-1) + 1\right] +$$
$$[(n+1)^2 - \left(\frac{t+1-1}{2}\right)(2n-(t+1)+4) - 2(z-1) - 1]$$

$$= n^2 + t - 1 < n^2 + 2n + 1 = (n+1)^2$$

If t is even and e as in (3) above then

$$S(e) = \left[\left(\frac{t}{2}-1\right)(2n-t+3) + 1 + 2(z-1) + 2\right] +$$
$$[(n+1)^2 - \left(\frac{t+1-1}{2}\right)(2n-(t+1)+4) - 2(z-1) - 2]$$

$= n^2 + t - 1 < n^2 + 2n + 1 = (n+1)^2$

Case 2) If $e \in B_t$, then we have

$$e = H(n+1,t)H(n+1,t+1) \qquad \dots(4)$$

Or

$$e = H(n+1, 2(n+1) - t)H\big(n+1, 2(n+1) - (t+1)\big) \qquad \dots(5)$$

If t is odd and e as in (4) above then

$$S(e) = \left[(n+1)^2 - \left(\frac{t-1}{2}\right)(2n - t + 4) - 2(n - t + 1 - 1) - 1\right] +$$

$$[\left(\frac{t+1}{2} - 1\right)(2n - (t+1) + 3) + 1 + 2(n - t - 1) + 1]$$

$$= (n+1)^2 - t < (n+1)^2$$

If t is odd and e as in (5) as above then

$$S(e) = \left[(n+1)^2 - \left(\frac{t-1}{2}\right)(2n - t + 4) - 2(n - t + 1 - 1) - 2\right] +$$

$$[\left(\tfrac{t+1}{2} - 1\right)(2n - (t+1) + 3) + 1 + 2(n - t - 1) + 2]$$

$$= (n+1)^2 - t < (n+1)^2$$

If t is even and e as in (4) as above then

$$S(e) = \left[\left(\frac{t}{2} - 1\right)(2n - t + 3) + 1 + 2(n + 1 - t - 1) + 1\right] +$$

$$[(n+1)^2 - \left(\tfrac{t+1-1}{2}\right)(2n - (t+1) + 4) - 2(n - t - 1) - 1]$$

$$= (n+1)^2 - (2n - t) < (n+1)^2$$

If t is even and e as in (5) above then

$$S(e) = \left[\left(\frac{t}{2} - 1\right)(2n - t + 3) + 1 + 2(n + 1 - t - 1) + 2\right] +$$

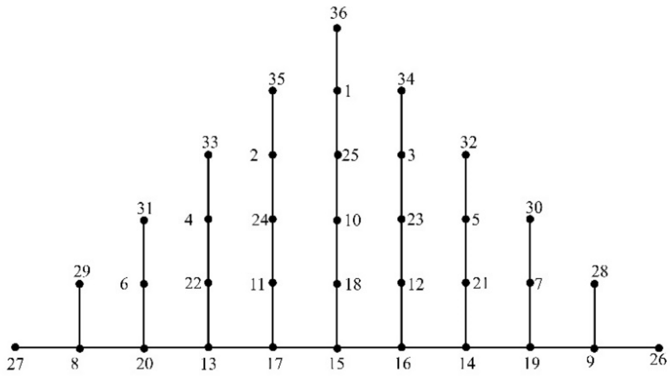
$$[(n+1)^2 - \left(\tfrac{t+1-1}{2}\right)(2n - (t+1) + 4) - 2(n - t - 1) - 2]$$

$$= (n+1)^2 - (2n - t) < (n+1)^2$$

So now we have labeled vertices of $TT(n+1)$ using the numbers from 1 to $(n+1)^2$ such that $S(e) \le (n+1)^2$ except for $e \in E_1$. As discussed above it can be seen that $S(e) = (n+1)^2 + 1$ hence $TT(n+1)\ U\ \overline{K_1}$ is 1 sw- sum graph when the isolated vertex is labeled as $(n+1)^2 + 1$ ■

An example of 1 sw- sum labeling of $TT(6)$ is shown below.

n=5 (odd)



TT(n+1)

Theorem 2: For any even $n, TT(n+1)$ is 1- optimal sw–sum graph.
Proof: Label the vertices of $TT(n+1)$ as follows.
For odd values of t and $1 \leq t \leq n+1$
Define

$$L(H(t,t)) = (n+1)^2 - \left(\frac{t-1}{2}\right)(2n-t+4)$$

$$L(H(t+z,t)) = (n+1)^2 - \left(\frac{t-1}{2}\right)(2n-t+4) - 2(z-1) - 1$$

$$L(H(t+z, 2(t+z)-t)) = (n+1)^2 - \left(\frac{t-1}{2}\right)(2n-t+4) - 2(z-1) - 2$$

where $z = 1$ to $(n+1) - t$
For even values of t and $2 \leq t \leq n$

$$L(H(t,t)) = \left(\frac{t}{2} - 1\right)(2n-t+3) + 1$$

$$L(H(t+z,t)) = \left(\frac{t}{2} - 1\right)(2n-t+3) + 1 + 2(z-1) + 1$$

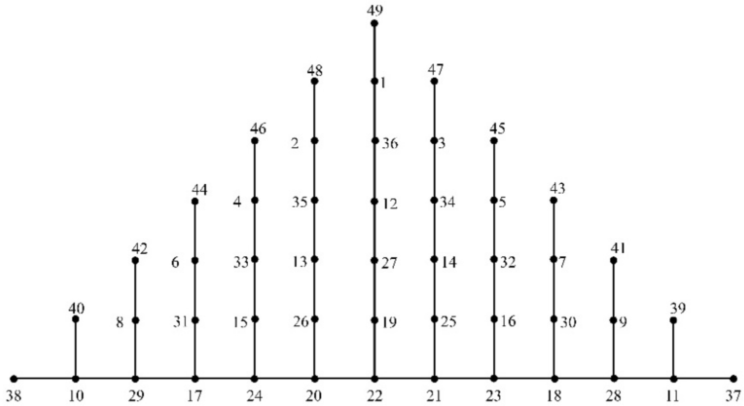
$$L(H(t+z, 2(t+z)-t)) = \left(\frac{t}{2} - 1\right)(2n-t+3) + 1 + 2(z-1) + 2$$

where $z = 1$ to $(n+1) - t$.

For any e$\in E(TT(n+1))$ it can be seen that $S(e) \leq (n+1)^2 + 1$

The proof is analogous to the theorem 1.■

An example of 1 sw- sum labeling of $TT(6)$ is shown below.



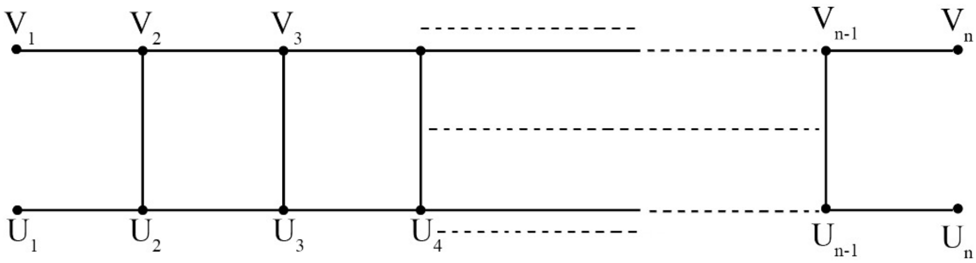
TT(n+1)

Theorem 3: $TT(n+1)$ is $\delta-$ optimal sw-sum graph.

Proof: It directly follows from theorem 1 and 2■

3. sw-sum labeling of Ladder graphs (L_n)

For $n > 2$, the ladder graphs on $2n$ vertices is denoted by L_n and has $3n-2$ vertices.The following is the example of ladder graph and the vertices are also named and the same style of naming vertices can be extended for bigger ladders.

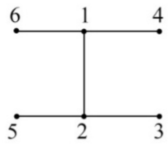


(L_n)

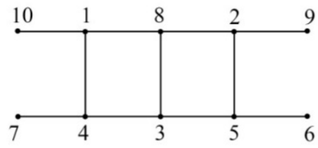
Now we will discuss the sw-sum optimality of L_n.

Theorem 4: For $3 \le n \le 6$, L_n is 1- optimal sw-sum graphs.

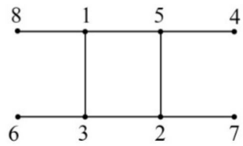
Proof: The following are 1- optimal sw-sum labels for L_3, L_4, L_5, L_6 and hence the proof.■



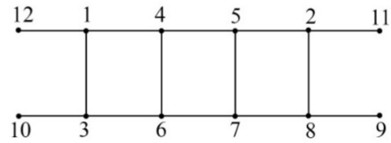
(L_3)



(L_5)



(L_4)



(L_6)

Theorem 5: For $n \geq 7$, L_n is not 1-optimal sw-sum graph.

Proof: We will prove that achieving 1-optimal sw-sum labeling of L_n is not possible. Consider L_n where $n \geq 7$. If we use the label $2n$ at any non-pendant vertex V then the at least one sum at the edge among three edges formed by the three adjacent vertices to V must be $2n + z$ where $z > 1$ then 1-optimal sw-sum labeling of L_n can't be achieved. So, to attempt 1- optimal sw-sum, we label any pendant vertex as 2n then the adjacent vertex to this vertex should be label as 1 is obvious. Now if we label the adjacent vertex of label receiving 1 as $2n - 1$, at least one of the remaining two edges at which the vertex labeled $2n - 1$ is one of its ends have the sum $2n - 1 + z$ where $z > 2$ will not allow us to achieve 1-optimal sw-sum. If $2n - 1$ label is assigned to any non-adjacent vertex of the vertex receiving label 1, then by the similar argument to the last statement we cannot achieve 1-optimal sw-sum. Continuing in the similar way we can see that in the attempt of achieving 1-optimal sw-sum labeling of L_n, the labels $2n, 2n - 1, 2n - 2, 2n - 3$ must be assigned at the pendent vertex and the adjacent vertices to these vertices should be assigned the labels 1, 2, 3, 4 so that the sums at the respective edges must be $2n + 1$. Now if we assign $2n - 4$ to any vertex which is adjacent to the vertex receiving one of the labels $1, 2, 3, 4$ then the sum at the edge between one of the other two adjacent edges must be $2n - 4 + z$ where $z > 5$. The same will happen if the label $2n - 4$ is assigned to any vertex which is not adjacent to the vertex receiving one of the labels $1, 2, 3, 4$. Hence there is no choice to use the label $2n - 4$ by which 1- optimal sw-sum of L_n can be obtained. ■

Theorem 6: The class L_n is not 1-optimal sw-sum graphs.

Proof: It directly follows from the theorem 5.■

Theorem 7: For all odd n such that $n \geq 7, w(L_n) = 2$

Proof: Label L_n as follows.

$L(U_1) = n, L(V_1) = n + 1$

For even t such that $0 \leq t \leq n-3$

$L(V_{n-t}) = 2n - t$

$L(U_{n-t-1}) = 2n - t - 1$

$L(U_{n-t}) = t + 1$

$L(V_{n-t-1}) = t + 2$

We call the steps of the ladder as $P_i = V_{i+1}U_{i+1}$ where $1 \leq i \leq n-2$

Now $S(P_i) = L(V_{i+1}) + L(U_{i+1})$

$S(P_i) = L\left(V_{n-(n-(i+1))}\right) + L\left(U_{n-(n-(i+1))}\right)$

$= 2n - \big(n - (i+1)\big) + \big(n - (i+1)\big) + 1$if i is even,

$S(P_i) = L\big(V_{n-(n-(i+2))-1)}\big) + L\big(U_{n-(n-(i+2))-1)}\big)$

$= \big(n - (i+2)\big) + 2 + 2n - \big(n - (i+2)\big) - 1$if i is odd.

Therefore $S(P_i) = 2n + 1$ for all i = 1 to (n-2)

Partition the edges of the type $\{V_jV_{j+1}\}$ for $j = 1\ to\ (n-1)$ as follows.

$\{V_1V_2\}, \{V_2V_3\}, \{V_{n-t}V_{n-t-1}, V_{n-t-1}V_{n-t-2} : t\ is\ even\ and\ 0 \leq t \leq n-5\}$

Now $S(V_1V_2) = L(V_1) + L(V_2) = L(V_1) + L\big(V_{n-(n-3)-1}\big)$

$= (n+1) + (n-3) + 2 = 2n$

$S(V_2V_3) = L(V_2) + L(V_3) = L\big(V_{n-(n-3)-1}\big) + L\big(V_{n-(n-3)}\big)$

$= \big((n-3)+2\big) + \big(2n - (n-3)\big) = 2n + 2$

For any other value of t,

$S(V_{n-t}V_{n-t-1}) = L(V_{n-t}) + L(V_{n-t-1}) = (2n - t) + (t+2) = 2n + 2$

$S(V_{n-t-1}V_{n-t-2}) = L(V_{n-t-1}) + L(V_{n-t-2}) = L(V_{n-t-1}) + L\big(V_{n-(t+2)}\big)$

$= (t+2) + 2n - (t+2) = 2n$

Partition the edges of the type $\{U_jU_{j+1}\}$ for $j = 1\ to\ (n-1)$as follows.

$\{U_1U_2\} \cup \{U_2U_3\} \cup \{U_{n-t}U_{n-t-1}, U_{n-t-1}U_{n-t-2} : t\ is\ even\ and\ 0 \leq t \leq n-5\}$

Now $S(U_1U_2) = L(U_1) + L(U_2) = L(U_1) + L\big(U_{n-(n-3)-1}\big)$

$= n + (2n - (n-3) - 1) = 2n + 2$

$S(U_2U_3) = L(V_2) + L(V_3) = L\big(V_{n-(n-3)-1}\big) + L\big(V_{n-(n-3)}\big)$

$= (2n - (n-3) - 1) + \big((n-3) + 1\big) = 2n$

$S(U_{n-t}U_{n-t-1}) = L(U_{n-t}) + L(U_{n-t-1}) = (t+1) + (2n - t - 1) = 2n$

$S(U_{n-t-1}U_{n-t-2}) = L(U_{n-t-1}) + L(U_{n-t-2}) = L(U_{n-t-1}) + L\big(U_{n-(t+2)}\big)$

$$= (2n - t - 1) + (t + 2) + 1 = 2n + 2.$$

Therefore, for every edge e of L_n, we have $S(e) \leq 2n + 2$.

Hence $L_n U \overline{K_2}$ has sw- sum labeling where the two isolated vertices added are labeled as $2n + 1$and $2n + 2$. From the theorem 5 it is clear that $w(L_n) \neq 1$. Therefore $1 < w(L_n) \leq 2$ implies $w(L_n) = 2$.■

Theorem 8: For all even n such that $n \geq 7, w(L_n) = 2$

Proof: Label L_n as follows.

For even t such that $0 \leq t \leq n - 2$

$L(V_{n-t}) = 2n - t$

$L(U_{n-t-1}) = 2n - t - 1$

$L(U_{n-t}) = t + 1$

$L(V_{n-t-1}) = t + 2$

The proof is analogous to theorem 7.■

Theorem 9: For all natural numbers n such that $n \geq 7$, $w(L_n) = 2$

Proof: It follows from theorem 7 and theorem 8.■

Future Scope

Triangular trees TT(n+1) can be considered for more labels.

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