Fixed Points of Weakly Contractive Self-Maps in Complete Metric Space

Dolhare U. P.[a], Bele C.D.[b]
[a] Department of Mathematics, D. S. M. College, Jintur. MS, India.
[b]Department of Mathematics, Shri Shivaji College, Parbhani MS, India.

Abstract

We prove some fixed point results for self-mappings satisfying weakly contractive conditions with fixed point results involving altering distance in complete metric space. Also the uniqueness of such fixed point are proved. In this paper we extend and generalize the results obtained by K. Goebel and W.A. Kirk and prove fixed point theorem for self maps.

1. **Introduction :**

Study of fixed points of self mappings satisfying contractive conditions which is one of the research activity. Large number of fixed point results for self-mappings satisfying various types of contractive inequalities which is in [7,10,15]. Fixed point results involving altering distances have been introduced in [12] . In 1977 Rhoades [2] proved fixed points for extended forms of contraction pairs. In 1986 Jungck [3] introduced the notion of compatible mappings, in 1998 Rhoades and Jungck [4] introduced the concept of weakly compatible maps. K. Goebel and W. A. Kirk [10] in 1990, V. I. Istratescu [15] in 1981 and M. S. Khan [12] in 1984 proved fixed point results for altering distances. In 1992 J. Meszaros [7] , In 1998 K. P. R. Sastry and G. V. R. Babu [8] proved fixed point theorems in Metric spaces by altering distances between the points. Naidu in 2003, Singh and Dimri in 2011 proved fixed point theorems for altering distance function. In 1997 Ya. I. Alber & Guerre [16] proved fixed point theorems for weak contraction. In 2011 Popescu proved fixed point theorems which involving weak contractive type inequalities and weak contraction. In 2013 Gairola and Ram krishan [14] ,A.Singh and R.C.Dimri [1], Lj. B. Ciric [11] , Naidu S.V.R. [13] proved some fixed point results for self-maps satisfying a generalized weak contraction conditions.

In this paper we introduce the generalized altering distance function and prove fixed point theorems for to obtain unique fixed point.

2. **Preliminaries and Definitions:**

Fixed point results involving altering distance have been introduced in [12]. An altering distance is a mapping,

$F : [\,0, \infty\,) \rightarrow [\,0, \infty\,)$ which satisfies,

(a) F is increasing and continuous and

(b) $F(t) = 0$ if and only if $t = 0$

Fixed point result for altering distances have also studied in [8, 9]
In this paper we obtain a new fixed point result for self-mappings defined on complete metric space which satisfying a contractive conditions which involves a function of two variables and acts on distances of two pair of points in a Metric spaces.

Definition 2.1 : A function $\phi : R \times R \rightarrow R$ is said to satisfy condition (a) if

(i) *ϕ is monotonic increasing in both the conditions*

(*ii*) *ϕ is continuous*

(*iii*) *$\phi(0,0) = 0$ and $\phi(\in,0) = 0$ implies $\in = 0$*

Let $\phi(\in,\in) = 0$. Then $\phi(\in,0) \leq \phi(\in,\in) = 0$ or $\phi(\in,0) = 0$ which implies that $\in = 0$ by (iii)

In this paper we state some important definitions which are useful for proving fixed point theorems.

Definition : 2.2 : Jungck [3] : Let S and T be mappings from a Metric space (X, d) into itself. The mapping S and T are said to be compatible if,

$\lim_{n\to\infty} d(ST_{x_n}, TS_{x_n}) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} S_{x_n} = t = \lim_{n\to\infty} T_{x_n}$ *for some* $t \in X$.

Definition : 2.3 : Jungck and Rhoades [4] : Let S and T be mappings from a Metric space (X, d) into itself. The mappings S and T are said to be weakly compatible if, they commute at their coincidence points, that is, if $Tu = Su$ for some $u \in X$, than $TS_u = ST_u$. in this connection if we write $v = Tu = Su$. Then we say that v is a point of coincidence of (S, T) .

Definition :2.4:[Khan, 1998][12] : A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called an **altering distance** function, if the following properties are satisfied.

(i) ψ is monotonically increasing and continuous

(ii) $\psi(t) = 0$ if and only if $t = 0$

Definition : 2.5 : [Rhoades, 2001] [2] : A mapping $T : X \rightarrow X$, where (X, d) is a Metric space, is said to be weakly contractive for $x, y \in X$,

$d(Tx, Ty) \leq d(x,y) - \phi(d(x,y))$

where $\phi : [o, \infty) \rightarrow [o, \infty)$ is continuous non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$. If $\phi(t) = (1-k)t$, where $0 < k < 1$, a weak contraction reduces to a Banach contraction.

Definition : 2.6 : [Choudhary, 2011] : Let (X, d) be a metric space, T a self mapping of X. We shall call T a generalized weakly contractive mapping if for all $x, y \in X$

$\psi(d(T_x,T_y)) \leq \psi(m(x,y) - \psi(\max\{d(x,y), d(x,T_x), d(y,T_y)\})$

where $m(x,y) = \max\{d(x,y), d(x,T_x), d(y,T_y), \frac{1}{2}[d(x,T_y) + d(y,T_x)]\}$

ψ is an altering distance function and

$\psi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function with ψ

3. Some fixed point theorems on self-map :

Theorem 1 : [Jaggi D. S.] [6] : Let f be a continuous, $f(t) = 0$ if and only if $t = 0$ self-map defined on a complete metric space (X, d). Further, Let f satisfy the following condition.

$$d(f(x), f(y)) \leq \frac{\propto d(x,f(x))\, d(y,f(y))}{d(x,y)} + \beta\, d(x, y)$$

for all $x, y \in X$, $x \neq y$ and for some $\propto, \beta \in [0, 1)$ with $\propto + \beta < 1$, then f has a unique fixed point in X.

D.S. Jaggi generalized theorem 1 for some integer m as follows.

Theorem 2 : [Jaggi D. S.] [6] : Let f be a self-map define on a complete metric space (X, d) such that for some positive integer m, f satisfy the condition

$d(f^m(x), f^m(y)) \leq \frac{\propto d\left(x, f^m(x)\right). \, d(y, f^m(y))}{d(x,y)} + \beta d(x,y)$ for all $x, y \in X$, $x \neq y$ and for some $\propto, \beta \in [0, 1)$ with $\propto + \beta < 1$. If f^m is continuous then f has a unique fixed point.
Jungck proved the following theorem for f–contractive point- to- point mapping for fixed point .

Theorem 3 : [Jungck] [5] : Let X be a complete metric space. Let f and g be commuting continuous self-maps on X such that $g(x) \subset f(x)$ further, let there exist a constant $\propto \in (0,1)$ such that for every x , y in X

$$d(g_x, g_y) \leq \propto d(f_x, f_y)$$

then f and g have a unique common fixed point.

Theorem 4 : [Lj. B. Ciric] [11] : Let X be a complete metric space. Let f be a self-map on X such that for some constant $\propto \in (0,1)$ and for every x, y in X

$$d(f_x, f_y) \leq \propto \max\{d(x,y), d(x,f_x), d(y, f_y), d(x, f_y), d(y,f_x)\}$$

Then f possesses a unique fixed point.

Theorem 5 : [K. Goebel and W. A. Kirk] [10] Let $F : X \to X$ be a self – mapping from a complete Metric space to itself and satisfy

$$[(\delta(f_x, f_y)^p + r(\delta(x, f_x))^q]^k + [(\delta(y, f_y))^p + r(\delta(y, f^2{}_x))^q]^k$$

$$\leq \lambda [(\delta(x,y))^p + r(\delta(x, f_x))^q]^k + \lambda'[(\delta(y,f_y))^P + r(\delta(y, f_y))^q]^k.$$

where $x, y \in X$; $p, k > 0$; $r, q \geq 0$ and $0 < \lambda < 1$; $0 < \lambda' \leq 1$, then f has a unique fixed point.

Main Result for fixed points

We generalized Theorem (5) by taking small values of p, q, r and k. consider $p = q = r = k = 1$ then we generalized theorem 5 as follows.

Theorem : Let $F : X \to X$ be a self-mapping from a complete metric space X to itself which satisfies the following inequality.

$$\phi(\delta(f_x, f_y), \delta(x, f_x)) + \phi(\delta(y, f_y), \delta(y, f^2{}_x))$$

$$\leq \lambda \phi(\delta(x,y), \delta(x, f_x)) + \lambda' \phi(\delta(y, f_y), \delta(y. f_x)) \qquad (1)$$

where $0 < \lambda < 1$; $0 < \lambda' \leq 1$; $x, y \in X$

and ϕ satisfies the condition (a) then f has a unique fixed point.

Proof : Let us consider $x_0 \in X$ and consider $\{x_n\}$ by

$$x_n = f_{x_{n-1}} = f^n x_o \quad \text{where} \quad n = 1,2,\ldots\ldots \qquad (2)$$

putting $y = f_x$ in (1) then we have

$\phi(\delta(f_x, f^2_x), \delta(x, f_x)) + \phi(\delta(f_x, f^2_x), \delta(f_x, f^2_x))$

$\leq \lambda \phi(\delta(x, f_x)), \delta(x, f_x)) + \lambda \phi(\delta(f_x, f^2_x), \delta(f_x, f_x))$ (3)

since $0 < \lambda' < 1$ and ϕ satisfies the condition (ii) in definition 2.1 then we have

$\lambda' \phi(\delta(f_x, f^2_x), 0) \leq \lambda' \phi(\delta(f_x,f^2_x), \delta(f_x,f^2_x))$

$\leq \phi(\delta(f_x,f^2_x), \delta(f_x, f^2_x))$ and from equation (3)

$\phi(\delta(f_x, f^2_x), \delta(x, f_x)) \leq \lambda \phi(\delta(x, f_x), \delta(x, f_x))$

$\leq \phi(\delta(x, f_x), \delta(x, f_x))$ (4)

which implies

$\delta(f_x, f^2_x) \leq \delta(x,f_x)$ (5)

By putting $x = x_{n-1}$ then we have

$o \leq \delta(x_{n+1}, x_n) \leq \delta(x_n, x_{n-1})$, $n = 1,2 \ldots\ldots$

This shows that $\{\delta(x_n, x_{n+1})\}$ converges

Let $\delta(x_n, x_{n+1}) = \ell$ (say) (6)
then from equation (4) by putting $x = x_{n+1}$ then we have
$\phi(\delta(x_n, x_{n+1}), \delta(x_{n-1},x_n)) \leq \lambda \phi(\delta(x_{n-1}, x_n), \delta(x_{n-1}, x_n))$ (7)
as $n \to \infty$ & ϕ is continuous then
$\phi(\ell, \ell) \leq \lambda \phi(\ell, \ell)$
or $\phi(\ell, \ell) = 0$ which implies that $\ell = 0$ by equation (2)
$\lim_{x \to \infty} \delta(x_n, x_{n+1}) = \ell$ (8)
If there exist $\in > 0$, and consider the subsequences $\{x_{m(\ell)}\}$ and $\{x_{n(\ell)}\}$ of $\{x_n\}$ such that for $m(\ell) < n(\ell)$
$\therefore \delta(x_{n(\ell)}, x_{m(\ell)}) \geq \in$ (9)
& $\delta(x_{n(\ell)-1}, x_{m(\ell)}) < \in$ then we have
$\in \leq \delta(x_{n(\ell)}, x_{m(\ell)}) \leq \delta(x_{n(\ell)}, x_{n(\ell)-1}) + \delta(x_{n(\ell)-1}, x_{m(\ell)}) < \delta(x_{n(\ell)}, x_{n(\ell)-1}) + \in$
taking $\ell \to \infty$ and using equation (8)
$\lim_{\ell\to\infty} \delta(x_{n(\ell)}, x_{m(\ell)}) = \in$ (10)
Also $\delta(x_{n(\ell)}, x_{m(\ell)}) \leq \delta(x_{n(\ell)}, x_{n(\ell)-1}) + \delta(x_{n(\ell)-1}, x_{m(\ell)-1}) + \delta(x_{m(\ell)-1}, x_{m(\ell)})$
and $\delta(x_{n(\ell)-1}, x_{m(\ell)-1}) \leq \delta(x_{n(\ell)-1}, x_{n(\ell)}) + \delta(x_{n(\ell)}, x_{m(\ell)}) + \delta(x_{m(\ell)}, x_{m(\ell)-1})$
as $\ell \to \infty$ and using equation (8) and (10) we have,
$\lim_{\ell\to\infty} \delta(x_{n(\ell)-1}, x_{m(\ell)-1}) = \in$ (11)
Also $\delta(x_{m(\ell)-1}, x_{n(\ell)+1}) \leq \delta(x_{m(\ell)-1}, x_{m(\ell)}) + \delta(x_{m(\ell)}, x_{n(\ell)}) + \delta(x_{n(\ell)}, x_{n(\ell)+1})$
and $\delta(x_{m(\ell)}, x_{n(\ell)}) \leq \delta(x_{m(\ell)}, x_{m(\ell)-1}) + \delta(x_{m(\ell)-1}, x_{n(\ell)+1}) + \delta(x_{n(\ell)+1}, x_{n(\ell)})$
taking $\ell \to \infty$ and using (8) and (10) then we have
$\lim_{\ell\to\infty} \delta(x_{n(\ell)-1}, x_{m(\ell)+1}) = \in$ (12)
Ingeneral $\delta(x_{n(\ell)}, x_{m(\ell)}) \leq \delta(x_{n(\ell)}, x_{m(\ell)-1}) + \delta(x_{m(\ell)-1}, x_{m(\ell)})$
and $\delta(x_{n(\ell)}, x_{m(\ell)-1}) \leq \delta(x_{n(\ell)}, x_{m(\ell)}) + \delta(x_{m(\ell)}, x_{m(\ell)-1})$

As $\ell \to \infty$ and using equation (8) and (10) we obtain

$\lim_{\ell\to\infty} \delta\left(x_{n(\ell)}, x_{m(\ell)-1} \right) = \in$

As $\ell \to \infty$, and consider equation (8), $x_n \to x$

and by continuity of ϕ then we have

$\phi(\delta(x, f_x), 0) + \phi(\delta(x, f_x), 0) \leq \lambda \phi (0, 0) + \lambda \phi (\delta(x, f_x), 0)$

which implies

$\phi(\delta(x, f_x), 0) \leq \lambda \phi (0, 0)$ in this case $0 < \lambda' < 1$

$\leq \lambda \phi (\delta(x, f_x), 0)$ using condition (ii) in definition (2.1)

Similarly $\phi(\delta(x,f_x), 0) = 0, \quad 0 < \lambda < 1$

so that $\delta(x,f_x) = 0$ [by condition (iii) in definition (2.1)]

which implies that $f_x = x$

this shows that f as a unique fixed point.

References :

[1] **A. Singh and R. C. Dimri** (2011) "*A common Fixed point through GeneralizedAltering Distance functions*" Kochi Journal of Mathematics, Vol. 6 PP 149-157.

[2] **B.E. Rhoades** (2001) "*Some theorems on weakly contractive maps*" non-linear Anal. TMA. Vol. 47, No. 4, pp 2683-2693.

[3] **G. Jungck**(1986)"*Compatible mapping and common fixed points*" Int. Indian Math and Math Sci, vol. 9, pp. 771-779.

[4] **G. Jungck and B. E. Rhoades** (1998) " *Fixed point for set valued function without continuity*" International Journal of pure and applied Mathematics, vol. 29, No3, pp-381-386.

[5] **Jungck G.** "*Commuting mappings and fixed points*" Amer. Math Monthly 83 (1976), 261 - 263.

[6] **Jaggi D.S.** "*Some unique fixed point theorems*".

[7] **J. Meszaros** "*A comparison of various definition of contractive type mappings*" Bull. Cal. Math. Soc. 84 (1992). 197-194.

[8] **K. P. R. Sastry and G. V. R. Babu**. " *Fixed point theorems in Metric spaces by altering distances*", Bull cal. Math. soc. 90 (1998), pp 175-182.

[9] **K.P.R. Sastry and G.V.R. Babu** " *Some fixed point theorems by altering distances between the points*", Indian J. Pure Appl. Math. 30 (1999), pp 641-647.

[10] **K. Goebel and W. A. Kirk** "*Topics in Metric fixed point theory* " Cambridge Uni press Cambridge, 1990.

[11] **Lj. B. Ciric** "*A generalization of Banach's contraction principle* " Proc. Amer. Math. Soc. 45 (1974), pp 267-273.

[12] **M.S. Khan. M. Swaleh and S. Sessa** (1984) " *Fixed point theorem by Altering Distance between the points*", Bulletin Australian Mathematical Society, Vol. 30 pp 1- 9.

[13] **Naidu S.V.R.** (2003) " *Some fixed point theorems in Metric space by altering distances*" Czechoslovak mathematical Journal, vol. 53. No.1 pp 205 – 212.

[14] **U. C. Gairola and Ram Krishan_** [2013] " *Common fixed point theorem for three maps by Altering Distances between the points* " IJMA, Vol. 4 No. 1, pp 143 – 151.

[15] **V. I. Istratescu**. " *Fixed point theory* " D. Riedel publishing co. London, 1981.

[16] **Ya. I Alber and S.D. Guerre** (1997). " *Principle of weakly contractive maps in Hilbert space* " New Results in operation theory, Editors I. Goldberg, Yu, advance and Appl. Vol. 98, Birkhaser verlog pp 7-22.