Children's understanding of Geometrical Concepts of 'Triangle' & 'Circle': Implications for Teaching Geometry

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Abstract

This article attempts to suggest the implications for teaching- learning based upon the findings of my research work. The article constitutes the part my M.Phil. dissertation that investigated the developmental changes in conceptual understanding of the concepts of 'triangle' and 'circle' of the students of grades V and VII. The analysis is based upon the theoretical framework of the Houdement and kuzniak's geometrical paradigms (2003), Fischbein theory of figural concepts (1993) and the mathematical meaning of 'Triangle'and 'Circle'. The findings suggest that there exist a gap between the personal and the mathematical meaning of these concepts and there was not many differences found in the understanding of the students of grade V and VII, in spite of beginning of the formal introduction of the geometry in the curriculum from grade VI.

KEYWORDS: figural concept, geometrical paradigms, grades V & VII, Triangle and Circle

Introduction

The 'Triangle' (Polygon) and 'Circle' are the geometrical figures that are presented to the children at early primary stage. In Indian schools, there are different ways how these concepts are introduced to the children at primary stage where geometry is not considered as the formal subject. In children's NCERT mathematics books of primary classes, the triangle is presented by showing three dimensional figures where the two-dimensional representation is triangular in form. However at upper primary level as geometry is introduced as formal subject, so an over-detailed explanation such as: "A triangle is a two-dimensional shape (a polygon) with three sides and three angles" is taught. Some teachers start with polygons, named according to the number of sides; a three-sided shape was named "triangle". Whereas some teachers introduced it by recognising different shapes and distinguish from other geometrical shapes. By the time children pass primary stage, mathematical object triangle become more sophisticated and properties, area and other concepts related to triangle are introduced assuming that children are well versed with the meaning of triangle as a mathematical object. Circle is informally introduced at class IV. It is usually introduced by showing wheels and bangles followed by explanation centre, radius and diameter and drawing of circle in text books. The main objective is to familiarise the children with the triangular and circular shapes around them. In Class VI it is again introduced with little bit more explanation as from class VI geometry is formally introduced. But definition of circle is not explained as such. However student must know the drawing circle using compass and freehand. They must also know its parts like diameter, radius and centre.

Vighi (2005) explored the progressive development of the concept of triangle in the minds of primary children. Findings revealed that most groups carried out classifications "incorrect" in Euclidean sense, but nonetheless based on well defined criteria. Many scholars (Duval, 1998; Parzysz, B. (1988); Berthelot & Salin, 1998)

identified several strong difficulties related to the use of figures in solving geometry problems at secondary school concerning both construction of figures and proofs of properties of geometric figures. Fischbein (1993) proposed the notion of 'figural concept' such that, while a geometrical figure such as a square can be described as having intrinsic conceptual properties (in that it is controlled by geometrical theory), it is not solely a concept, it is an image too (I*bid*, p141). Accordingly, geometrical reasoning is characterized by the interaction between these two aspects, the figural and the conceptual. Fischbein also suggested that "the process of building figural concepts in the students' mind should not be considered a spontaneous effect of usual geometry courses" (Fischbein, 1993). (Monaghan 2000) Monaghan discussed the way in which students perceive various quadrilaterals, and particularly the connection between the perception and description of a given geometrical figure.
Researchers have also shown that most of the difficulties are experienced by the students during the transition from elementary/primary to secondary school which is evident in their performance in most topics, especially in mathematics. It is considered to be as a critical life event, since progressing from one level of education to the next, students may experience major changes in school climate, educational practices, and social structures (Rice,2001). (Deliyianni, Elia & Gagatsis, 2009) investigated the role of various aspects of apprehension, i.e., perceptual, operative and discursive apprehension, in geometrical figure understanding of primary and secondary school student. Findings revealed differences between primary and secondary school students' performance and in the way they behaved during the solution of the tasks. (Panaoura & Gagatsis,2009) studied the geometrical reasoning of primary and secondary school students in order to investigate the strategies and common errors students make while transition from natural geometry(the objects of this paradigm of geometry are material objects) to Natural Axiomatic Geometry (definitions and axioms are necessary to create the objects in this paradigm of geometry). These findings stress the need for helping students progressively move from the geometry of observation to the geometry of deduction.

The present study

The basic purpose of the study was to explore the developmental changes in conceptual understanding of the geometrical concepts of 'Triangle' and 'Circle' among the children of grades V and VII. It was an attempt to look at perceptual, lexical and figural aspect (fischebein, 1993) among students of class V and VII and to see the conceptual changes from V to VII class students. The study also tried to explore the factors contributing to the development of the concept viz. Pedagogical Practices in mathematics, Support available at home regarding learning mathematics, Resources available at school for mathematics. An attempt was also made to suggest the implications of the findings for teaching of geometry.

Theoretical frameworks

Houdement and Kuzniak's Theoretical Perspective

The article focuses on the part of this theoretical perspective that considers geometry as a theory of space, which tends to represent the local properties of real space. Its more elaborate form is R3 with the structure of a Euclidean space. This perspective divides geometry into three paradigms viz**. Geometry I (Natural Geometry) which is related to reality. Perception, experiment and deduction are the means to act on the material objects. Geometry II (Natural Axiomatic Geometry) where**

hypothetical deductive laws in an axiomatic systems are to be used for validation , however relation to reality remains important. **Geometry III (Formalist Axiomatic Geometry),** the system of axioms is complete and independent of its possible applications to the world. The only criterion of truth is consistency (i.e. absence of contradictions). “Fundamental principle is that the various proposed paradigms are homogeneous: it is possible to reason inside one paradigm without knowing the nature of the other” (Houdement & Kuzniak, 2003).

Fischebein’s theory of figural concepts

In his paper, it is argued that geometrical figures are characterized by both conceptual and sensorial properties. A geometrical figure is a mental abstract, ideal entity, the meaning of which is governed by a definition. At the same time, it is an image: it possesses extensiveness (spatiality), shape and magnitude. In geometrical reasoning the two categories of properties should merge absolutely, with the sensorial components providing the dynamics of invention and the conceptual component guaranteeing the logical course of the mathematical process.

Triangle

What is a triangle? Is a triangle an abstraction of a concrete objects? In nature there are very few examples of triangles. If we try to find, in the manner of Galilean philosophy, correspondence between objects in nature and mathematical shapes, it is difficult to place the triangle. According to Plato, the triangle is an idea, which exists independently of human thought, and according to Aristotle a triangle is an imminent shape upon real objects. Euclid defines the rectilinear figures (Definition XIX) those bound by straight lines; trilateral shapes are bound by three straight lines. He classifies them according to sides or angles (Definitions XX and XXI).

Circle

The invention of the wheel is a fundamental discovery of properties of a circle. Mathematically, the circle is defined as the set of all points that are the same distance from a given point, called the center. That distance is called the radius. The word radius comes from the Latin word for rod or spoke of a wheel, and the radii, or spokes, radiate out from the center. The definition of a circle is the set of points all at the same distance from a given point called the centre. This definition can be used to draw a circle in the sand with a piece of string, or on paper with a compass. Our word center comes from the Greek word "kentron" meaning sharp point, as in the sharp point of the compass. Each point is equally distant from the axle of the wheel, making the tire roll smoothly. This makes the circle the ideal shape for gears and wheels and anything that rolls. The circle is the same on all sides. Designers choose this shape when they design an object that has no top or bottom, front or back, and can be used equally well from all sides.

Methodology

This study was done on 132 students of the three government schools of the capital (New Delhi) of India in two phases. Classes 5^{th} and 7^{th} were found to be appropriate as class 5^{th} being end of primary mathematics education, students of 5^{th} are familiar with most of geometrical concepts taught in an informal way without actually introducing geometry as subject and from class 6^{th}, geometry is introduced and taught in a formal way, class 7^{th} is first class after transition from primary to upper primary, to see the development and changes in the geometrical concepts class 7^{th} was selected.

The study was conducted in two phases. In Phase 1, In order to invoke what 'Triangle' and 'Circle' bring to mind of students, four activities (Drawing, Brainstorming, Defining, Identification) were prepared. (Based upon the study did by Vighi in 2005) .To explore the other factors which may contribute to the developmental changes in the mathematical concept, classroom observations, teachers' interview were conducted and information about the students' backgrounds were gathered. Selected students were of low economical status, Parents of most of the students were uneducated and most of the students were sent for mathematics tuition in their nearby place which were not of very good standard as tuition fee of these centers were very low(around Rs.150 to Rs. 200 for all subject).
In the second phase, in-depth interviews of the selected 20 students were conducted. These students were selected on the basis of their responses given in the activity sheets.

Description of data Gathering Resources
Activity Sheets for Students
Activity One: Drawing
Students were asked to draw 4 different triangles and circles, using a sheet of blank paper and they were free to draw either using geometrical tools or free hand.
Following tasks were carried out:

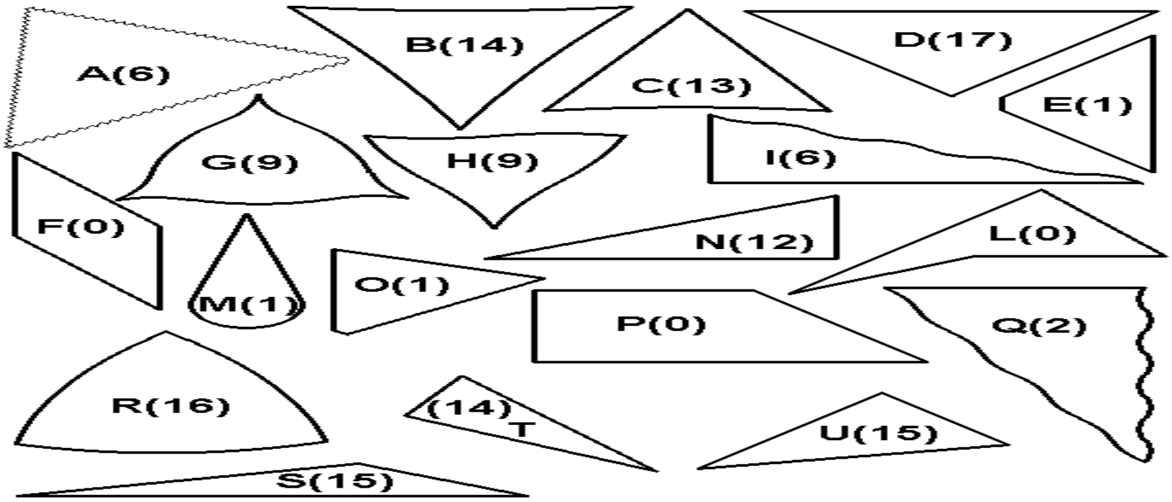
1. Draw a triangle/ Circle
2. Draw a different triangle/Circle
3. Draw another different triangle/ Circle
4. Draw another triangle different from all three.

The children were also asked to explain the choices they made in their drawing, and to explain the 'differences.

Activity Two: Brainstorming
In order to further investigate, brainstorming activity was carried out. Each child was asked about the think of a word Triangle/Circle and draws everything that the word suggested and they see around.

Activity Three: Defining
Children were asked to give written explanation of what they mean by 'Triangle' and Circle in their own words.

Activity Four: Identification
Children were asked to look at different shapes and their features, identify their main characteristics and decide whether those are triangle/Circle or non triangle/ non circle corresponded to the explicit or implicit definition of a triangle.



Class Observations

Selected section of classes 5th and 7th of selected three schools were observed for 2-2 days to get a feel about general pedagogical practices in mathematics followed in the class and to get familiar with the students. However, these class observations were not specific to geometry teaching but to observe the class environment, interaction between student and teacher, students' participation in the class and resources used by teacher in the class to teach mathematics.

Semi-Structured Interview for Teachers

Teachers were asked about general information viz. name, class taught academic qualification, and professional qualification experience and school name.

In the 2nd part of the interview teachers were asked about the following dimensions:

- The resources available in school for mathematics teaching
- Responses from parents of school children
- Books referred by teachers to teach mathematics
- Book followed in class to teach mathematics
- How do they introduce a chapter generally

Questionnaire for Students

There were two sections in the questionnaire along with the general information section. In general information section the student had to fill: name, class, age and school name. Sections were divided dimension wise and comprised both closed and open ended questions.

Phase 2 comprised in-depth interview of the students on the basis of the analysis of the responses given in the activity sheets of phase 1. 10 students of grade V and 10 students of grade VII were selected for the in-depth interview on the basis of variations in the answers.

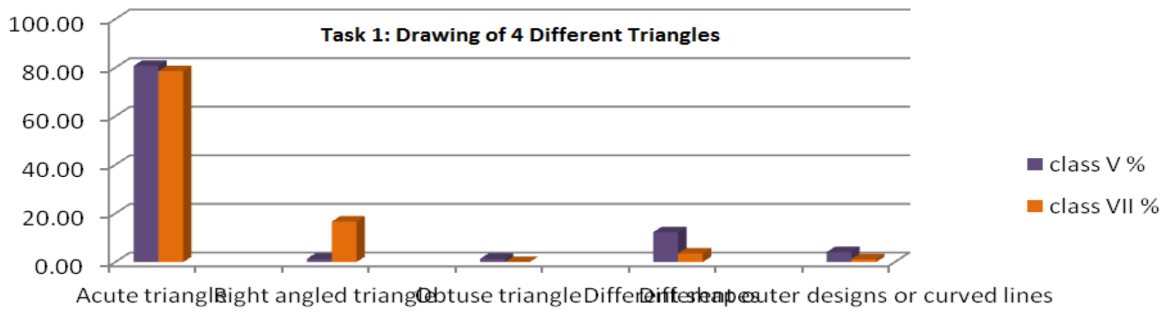
Results

Some of the major findings based upon four activities given to the students related to the Development of the concept Triangle are as following:

Most of the students gave explanation about the difference that was based upon perceptual aspect of geometrical Orientation i.e. vertex with changing directions E.g. **in the in-depth interview:**

Question: ***Aap ko ye triangles alag kyo lag rhe hai?***
Answere: ***Ek ki chonch upper hai, dusre ki neeche hai***

- Another significant aspect is that the need for diversification led a number of children to change the sides of the triangle. They thus obtained what they referred to as a "moved", 'inclined' triangle', or a triangle with curved sides or even, more surprising, a triangle where outer sides are designed. Thus many children do not have the idea of side or rather they have, a very specific idea, certainly not Euclidean.
- The most significant result however was that the triangle is essentially the same, equilateral and with one side horizontal. It is "the" triangle; some children call it the "normal/simple triangle". Students have a perception about triangle which should have a horizontal base and a point. The plausible explanation is that this type of triangle is usually drawn by their teacher or most of the time it is in their textbooks.



- Looking at the changes from class V to class VII there is not much difference found in the drawing of triangles as most of the students drew acute angled triangle which is considered as prototype (Hershkowitz, Vinner and Bruckheimer, 1987) of the triangle except very few students of class VII who drew right angled triangle as different one.

- Looking at the lexical aspects i.e. language used by students, it can be seen that more students of class VII used more propositions instead of using natural speech for the explanation given to make different choices e.g. students of class VII could differentiate in terms of types according to sides i.e. scalene, isosceles and equilateral but they could not use the same criteria for drawing activity so there is a plausible explanation that students tried to write the ratify answers which were taught in the class.

- Only 2% of class VII student could define triangle, but most of the students gave partially correct definition. Their explanation includes the following attributes :
  - It is a shape with three peaks(three angles).
  - Triangle has three sides.
  - It has three corners.
  - Another explanation is horizontal line and a peak.

  - Class VII students also tried to explain triangle by giving its types viz. scalene, equilateral and isosceles triangle and explaining them what they are.
  - One line horizontal and other two lines are inclined.

- Another aspect that was found that figure D was identified as a triangle by all the subjects questioned. This is, certainly, the prototype of the triangle concept (what may be called specifically an 'equilateral Triangle).

- Most of the class V students were not able to handle the compass to draw circle.

- Another significant aspect that emerges which is consistent with what Duval explains about three cognitive processes i.e. student could not draw equilateral triangle though he knows what it is.

Other interesting findings based upon the teachers' interview and other resources are:

- Primary school teachers are mother teachers and teach all the subjects so lack of expertise.
- There are no mathematics lab facilities available for the students of V and VII.
- No interaction between primary school teacher and upper primary school teacher about the pedagogical practices .
- Students belong to poor families and parents are not educated still they go to mathematics tuition;
- There is possibility that these tuitions are not of good standard, so students instead of understanding the concept tried to give memorized answers.

Major findings related to the Development of the concept 'Circle' are as following:

- Students have a vague idea about the word 'round', round for them means without any cut and sharp point or a round figure is one that does not have edges.
- Another important aspect that Circle is considered as round figure with no shape it means for them shape means is sharp pointed with proper edges.
- Students could not generate of the formal definition of circle, though it was not taught at that level. So it seems important in case of circle that formal definition of circle needs mediation.
- Class V students could not handle the compass to draw circle they either drew it free hand or with using any round shape. However class VII students used compass to draw circle.
- The another significant result that I circle drawn by most of the students was perfectly round but 2nd and 3rd circle was made either in oval shape or with different outer designs in order to make them different. I circle like triangle was considered normal circle. So the word 'different' made them think to distort the actual shape of circle.
- Looking at the changes from class V to class VII there is not much difference found in the drawing of circles however while explaining few students of class VII used the criteria of difference in radius. Not even a single child could give

a formal definition of circle so it shows importance of instruction in spite of age as factor for the development of the concept.

- As far as lexical aspect is concerned, students used explanation in natural speech e.g. circles are different because a) they are different in size b) they are different shapes like moon sun c) they are made using different instruments.

- Students are not able to define circle without being taught however they were able to identify it correctly, plausible explanation is that as circle is very close to real life objects so figural aspect is stronger than the conceptual one.

Discussion and Implications for Teaching and Learning Geometry

As far as the development of the understanding of these concepts is concerned, researcher found a huge gap between the personal and geometrical meaning of the concepts among students of class V and VII. Though there is an improvement in the answers of class VII students than class V, analysis of the data shows that instruction provided in the class may responsible for the quality of answers with regard to the control exerted by the conceptual constraints on the figural interpretation of geometrical entities as the attributes explained by the most of the students while defining 'Triangle' were not being used while identifying those figures. Analysis of the data also shows that class V and VII students still belong to Geometrical Paradigm 1 (Houdement and Kuzniak,2003), as responses of most of the students were based more on intuition and experience than on properties and definitions. Here to note that as per the three paradigms of geometry, students should be able to use geometrical instruments in paradigm 1, but most of the class V students were not able to handle the compass. Another significant aspect that emerges which is consistent with what **Duval** explains about three cognitive processes i.e. student could not draw equilateral triangle though he knows what it is. It means development of reasoning or visualization does not assure the development of construction.

As far as **implications** of the findings of the teaching of geometry are concerned, Teaching and learning geometry depends upon the epistemological nature and the cognitive processes involved in understanding geometry.

While discussing its epistemological nature-two aspects have been seen. On one hand, it is the study of logical relations, deductive reasoning and on the other hand it refers to spatial concepts, intuitive understanding.

> *The goal of geometry learning should be the realization of geometry as a deductive structure, with geometry as the science of our environment as a necessary pre-requisite. Findings of the study helps to* decide the extent of depth of mathematical knowledge and reasoning teachers need for teaching, Based upon these aspects and the cognitive process involved in the development of geometrical thinking, it can be inferred that primary teachers should be at least, in the second paradigm (geometry II), and secondary school teachers should be in third paradigm (geometry III) to help making the transition easier from one paradigm to 2^{nd} paradigm (Dorier, Gutierrez, Strasser, 2003)

. The van Hiele levels also provide a good reference to answer the question on reasoning: teachers should reason at least one level higher than students although they have to be aware that they should interact with their pupil on their level). Then primary school teachers should be atleast in the third level and secondary school teachers should be at least in the forth level. Therefore, teacher training courses should allow perspective teachers to learn the relationship among abstract geometrical concepts or properties and their concrete representations, and to discover geometric models in children's ordinary life environment.

- Teachers should focus on different kinds of triangles besides just introducing prototype of triangle. The process of construction and reasoning should be developed separately as they are independent of each other keeping in mind the synergy of these processes is necessary for proficiency in geometry. (Duval, 1998).
- There were some triangles(N,S,T,U) which most of the students could not identify as triangles, some of the reason s they gave are: N because its one side is standing straight and they consider a triangle with horizontal side and two inclined sides, S because it is touching horizontal; T because it is very thin, U earlier was not accepted by many students as they were thinking that it does not have horizontal base but later on in the in-depth interview almost all children accepted it. Teachers should focus on these types of triangles besides just introducing prototype of triangle.
- As the analysis shows that students marked those shape as triangle which are not triangle in the Euclidean space but could be triangle in other space like in non Euclidean space e.g. B and C are triangles in non Euclidean space; teachers needs have knowledge about the different type's geometry.
- Transition from geometry I to geometry II is important but crucial Therefore, teacher training courses should allow perspective teachers to learn the relationship among abstract geometrical concepts or properties and their concrete representations, and to discover geometric models in children's ordinary life environment.
- As Euclidean geometry is most prevalent at school level so it is essential for the teachers to know its triumphs and limitation. It will help to realize the objectives of teaching. At lower level we try to relate students with the physical reality, teachers must know that in Euclid geometry we use ideal object as approximation while applying to our space. It is just a matter of convenience. As recent developments in psychological view of geometry have also shown the importance of spatial reasoning, Euclidean geometry has not remained just a fixed body of axioms and postulates and deductive reasoning. Since then the content of this geometry has been revised by many modern mathematician.

- To understand the children's understanding of geometric thinking

Other theoretical frameworks (alternative routes by Battista and Houdements and kuzniak' theoretical framework)which are built up to elaborate or revise Van-Heile levels help to provide more elaborate understanding of children's understanding of geometry i.e. how does it develop in small incremental steps and progress through transition phases.

- To decide the appropriate tools for the learning of geometry

Van Hiele levels give importance to teaching and using instruments to develop geometrical reasoning as at level I, II, III the-re is need of physical objects, instruments and drawing to help students solve task or understand geometrical structure or organize their reasoning. Technology is also being used for the learning of geometry as it focus on the aspects of visualization and spatial reasoning which are important aspects of geometrical thinking. Laborde; Kynigos; Hollebrands; Strasser(2006) found dialectical link between the development of theories and research on the use of technology in geometry.

- Major implications of Duval's theoretical framework:

- The three kinds of processes i.e. visualisation, construction and reasoning must be developed separately. This framework says that they are independent of each other
- As there are various kind of reasoning and various ways to see a figure so it is important for teachers to use differentiated task for the students to develop various kinds of reasoning and visualization.
- Those task also be involved by which there processes can be used in co-ordination as "their synergy is cognitively necessary for proficiency in geometry" (Duval, 1998).

Some of the plausible explanations related to the pedagogical practices and lack resources related to mathematics teaching might be the factors responsible for lack of understanding of Geometry. Observation of class V and VII shows that Primary school teachers are mother teachers and teach all the subjects so lack of expertise. There are no mathematics lab facilities available for the students of V and VII. Another aspect is that most of the students belong to poor families and parents are not educated still they go to mathematics tuition; there is possibility that these tuitions are not of good standard, so students instead of understanding the concept tried to give memorized answers. But more investigation is required pertaining to the same.

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