Exact Solution of perfect fluid massive string cosmology in Five- dimensional Bianchi Type III space-time with decaying vacuum energy density Λ

R. K. Jumale, I. S. Mohurley[a], D. H. Gahane[b] & Jyotsna Jumale[c]
[a]Department of Physics, Shri Dnyanesh Mahavidyalaya, Nawargaon-441223, M.S., India
[b]Department of Physics,N. H. College, Bramhapuri, Distt- Chandrapur, India
[c]R.S.Bidkar College, Hinganghat, Distt. Wardha, India

Abstract

We have obtained exact solution of Einstein's field equations for massive string cosmological model of five dimensional Bianchi-III space time with decaying vacuum energy density Λ on the lines of Letelier (1983). We have assumed that the expansion θ in the model is proportional to the eigen value σ_3^3 of the shear tensor of σ_j^i to get the exact solution of five dimensional Einstein's field equations in which the fluid satisfies the baratropic equation of state. It has been observed that the vacuum energy density Λ is positive and decreasing function of time. Finally, we observed that the string dominates over the particle in the early stage of evolution of the universe and the universe in dominated by massive string throughout the whole process of evolution of our model in early stage as well as late time. We have discussed some important physical and geometric properties of the model of five dimensional Bianchi-III space time.

KEYWORDS : Exact solution of Einstein's field equations, Massive string, Perfect fluid vacuum energy density Λ**.**

§1. Introduction. Cosmic strings play a vital role in study of the early universe. The solution of the Einstein field equations has been obtained by Letelier (1979) for a cloud string with spherical, plane and cylindrical symmetry. Further, he has obtained cosmological model for a cloud of massive string in Bianchi type-I space-time. Also exact Bianchi-III cosmological solutions of massive strings in presence as well as absence of magnetic field have been obtained by Tikekar and Patel (1994), using the techniques of Letelier (1979,1983) and Stachel (1980). Wang (2003, 2004a, 2004b, 2005, 2006) has discussed LRS Bianchi-I and Bianchi-III cosmological model for a cloud string with bulk viscous fluid. Recently cosmic strings with bulk viscosity have been obtained by Tripathi et al. (2009, 2010). The string cosmological models in different contexts have been studied by Pradhan et al. (2008, 2009, 2010). Also they have obtained LRS Bianchi-II cosmological model with perfect fluid distribution of matter and string dust respectively.

In the paper [1], Pradhan et.al. (2011), have discussed the problem of establishing a formalism for study of the massive string in Bianchi-III space-time and they have investigated an exact and general solution for Bianchi-III cosmological model for cloud of a strings in four dimensional space-time. It has been observed that the study regarding Bianch-III cosmological model with decaying vacuum energy density Λ in four dimensional space-time can further be extended to higher five dimensional Bianchi-III space-time and therefore, an attempt has been made in the present paper.

§ 2 Five dimensional Einstein's field equations

We consider five dimensional Bianchi- III space-time with the line element

$$ds^2 = -dt^2 + A^2[dx^2 + dy^2] + B^2 e^{-2ax} dz^2 + C^2 du^2 , \quad (1)$$

where A, B, C are the functions of time t and a is constant. The energy momentum tensor for a cloud of string with a perfect fluid is given by

$$T_i^j = (\rho + p) v_i v^j + p g_i^j - \lambda x_i x^j , \quad (2)$$

where v_i and x_i satisfy condition

$$v^i v_i = -x^i x_i = -1 , \quad v^i x_i = 0 , \quad (3)$$

Here p is isotropic pressure, ρ is the proper energy density for a cloud of strings with particles attached to them, λ is the string tension density, v^i is the five-velocity of the particles and x^i is a unit space-like vector representing the direction of string. If the particle density of the configuration is denoted by ρ_p, then we have

$$\rho = \rho_p + \lambda . \quad (4)$$

The five dimensional Einstein's field equations are given by

$$R_i^j - \frac{1}{2} R g_i^j = 8\pi T_i^j - \Lambda g_i^j , \qquad (i, j = 1, 2, 3, 4, 5) \quad (5)$$

where R_i^j is the Ricci tensor; $R = g^{ij} R_{ij}$ is the Ricci scalar and Λ is cosmological constant.
In a co-moving co-ordinate system, we have

$$v^i = (0,0,0,0,1) , \quad x^i = (0,0,0,1/C,0) . \quad (6)$$

The field equations (5) together with (2) yield

$$R_1^1 - \frac{1}{2} R g_1^1 = 8\pi T_1^1 - \Lambda g_1^1 \Rightarrow \quad \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} = 8\pi p - \Lambda , \quad (7)$$

$$R_2^2 - \frac{1}{2} R g_2^2 = 8\pi T_2^2 - \Lambda g_2^2 \Rightarrow \quad \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{a^2}{A^2} = 8\pi p - \Lambda , \quad (8)$$

$$R_3^3-\frac{1}{2}Rg_3^3=8\pi T_3^3-\Lambda g_3^3 \;\Rightarrow\; \frac{2\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{2\dot{A}\dot{C}}{AC}+\frac{\dot{A}^2}{A^2}=8\pi p-\Lambda, \tag{9}$$

$$R_4^4-\frac{1}{2}Rg_4^4=8\pi T_4^4-\Lambda g_4^4 \Rightarrow \frac{2\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{2\dot{A}\dot{B}}{AB}-\frac{a^2}{A^2}+\frac{\dot{A}^2}{A^2}==8\pi(p+\lambda)-\Lambda, \tag{10}$$

$$R_5^5-\frac{1}{2}Rg_5^5=8\pi T_5^5-\Lambda g_5^5 \Rightarrow \frac{2\dot{A}\dot{B}}{AB}+\frac{2\dot{A}\dot{C}}{AC}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{A}^2}{A^2}-\frac{a^2}{A^2}=-8\pi\rho-\Lambda. \tag{11}$$

$$R_1^5-\frac{1}{2}Rg_1^5=8\pi T_1^5-\Lambda g_1^5 \;\Rightarrow\; a(\frac{\dot{A}}{A}-\frac{\dot{B}}{B})=0. \tag{12}$$

Here an over dot denotes ordinary differentiation with respect to t.

The spatial volume for our five dimensional model (1) is defined as

$$V^4=A^2BC \tag{13}$$

where $V=[A^2BC]^{\frac{1}{4}}$ is the average scale factor.

Therefore, the average Hubble's parameter of anisotropic models read as

$$H=\frac{\dot{V}}{V}=\frac{1}{4}[H_1+H_2+H_3+H_4]=\frac{1}{4}[\frac{2\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}]. \tag{14}$$

The directional Hubble's parameters along x, y, z and u are defined by

$$H_1=H_2=\frac{\dot{A}}{A}, \qquad H_3=\frac{\dot{B}}{B}, \qquad H_4=\frac{\dot{C}}{C}. \tag{15}$$

The deceleration parameter q is given by

$$q=-\frac{V\ddot{V}}{\dot{V}^2}. \tag{16}$$

The velocity field v^i as specified by (6) is irrotational. The scalar expansion θ, components of shear σ_{ij} and the average anisotropy parameter A_m are defined by

$$\theta = \frac{2\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}. \tag{17}$$

$$\sigma_{11} = \sigma_{22} = \frac{A^2}{4}\left[\frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right], \tag{18}$$

$$\sigma_{33} = \frac{B^2 e^{-2ax}}{4}\left[\frac{3\dot{B}}{B} - \frac{2\dot{A}}{A} - \frac{\dot{C}}{C}\right]. \tag{19}$$

$$\sigma_{44} = \frac{C^2}{4}\left[\frac{3\dot{C}}{C} - \frac{2\dot{A}}{A} - \frac{\dot{B}}{B}\right], \tag{20}$$

$$\sigma_{55} = 0. \tag{21}$$

Therefore, $$\sigma^2 = \frac{1}{4}\left[\frac{2\dot{A}^2}{A^2} + \frac{3}{2}\frac{\dot{B}^2}{B^2} + \frac{3}{2}\frac{\dot{C}^2}{C^2} - \frac{2\dot{A}\dot{B}}{AB} - \frac{2\dot{A}\dot{C}}{AC} - \frac{\dot{B}\dot{C}}{BC}\right]. \tag{22}$$

$$A_m = \frac{1}{4}\sum_{i=1}^{4}\left(\frac{\Delta H_i}{H}\right)^2, \tag{23}$$

where $\Delta H = H_i - H (i = 1,2,3,4)$.

All the results in five dimensional space time are in the format of Pradhan et al (2011).

§3.Solutions of the Einstein's field equations in V_5

The field equations (7)-(12) are a system of five equations with seven unknown parameters $A, B, C, \rho, p, \lambda$ and Λ.To get the explicit solution of system of the equations, we have added two additional constraints relating these parameters. We assume that the expansion (θ) in the model is proportional to the eigen value σ_3^3 of the shear tensor σ_i^j. This condition leads to

$$B = l_1 (A^2 C)^{m_1}, \tag{24}$$

where l_1 and m_1 are arbitrary constants. Equations (12) leads to

$$A = mB, \tag{25}$$

where m is an integration constant. Equations (7) and (9) reduce to

$$\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}-\frac{\dot{A}\dot{B}}{AB}-\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{A}^2}{A^2}=0 \tag{26}$$

Using (25) in (26), we get

$$(1-m)\left(\frac{\ddot{B}}{B}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{B}^2}{B^2}\right)=0. \tag{27}$$

As $m \neq 0$, (27) gives

$$\left(\frac{\ddot{B}}{B}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{B}^2}{B^2}\right)=0 \tag{28}$$

which on integration reduces to

$$B\dot{B}C=k_1, \tag{29}$$

where k_1 is an integration constant .

From (24) and (25), we obtain

$$B=l_2C^l, \tag{30}$$

where $l_2=l_1^{\frac{1}{1-2m_1}}m^{2l}$ and $l=\frac{m_1}{1-2m_1}$. Using (30) in (29) we get

$$C^{2l}\dot{C}=\frac{k_1}{ll_2^2}, \tag{31}$$

which on integration gives

$$C=[2l+1]^{\frac{1}{2l+1}}\left[\frac{k_1}{ll_2^2}t+k_2\right]^{\frac{1}{2l+1}}, \tag{32}$$

where k_2 is an integrating constant. Using (31) in (29) and (25) we obtain

$$B = l_2[2l+1]^{\frac{l}{2l+1}}\left[\frac{k_1}{ll_2^2}t + k_2\right]^{\frac{l}{2l+1}}, \tag{33}$$

and

$$A = ml_2[2l+1]^{\frac{l}{2l+1}}\left[\frac{k_1}{ll_2^2}t + k_2\right]^{\frac{l}{2l+1}}. \tag{34}$$

respectively, which are the required solutions of Einstein's field equations in V_5.

Hence the line element (1) becomes

$$ds^2 = -dt^2 + \{ml_2(2l+1)^{\frac{l}{2l+1}}[\frac{k_1}{ll_2^2}t + k_2]^{\frac{l}{2l+1}}\}^2(dx^2 + dy^2) + \{l_2(2l+1)^{\frac{l}{2l+1}}e^{-ax}[\frac{k_1}{ll_2^2}t + k_2]^{\frac{l}{2l+1}}\}^2 dz^2$$

$$+\{(2l+1)^{\frac{1}{2l+1}}[\frac{k_1}{ll_2^2}t + k_2]^{\frac{1}{2l+1}}\}^2 du^2. \tag{35}$$

To study the physical and geometric properties of the model, we use the transformation as under

$$ml_2(2l+1)^{\frac{l}{2l+1}}x = X, \qquad ml_2(2l+1)^{\frac{l}{2l+1}}y = Y, \qquad l_2(2l+1)^{\frac{l}{2l+1}}z = Z,$$

$$(2l+1)^{\frac{1}{2l+1}}u = U, \qquad \frac{k_1}{ll_2^2}t = T, \tag{36}$$

the metric (35) reduces to

$$ds^2 = -\beta^2 dT^2 + T^{2L}[dX^2 + dY^2] + T^{2L}e^{-\frac{2aX}{N}}dZ^2 + T^{\frac{2L}{l}}dU^2, \tag{37}$$

where $\beta = \frac{ll_2^2}{k_1}$, $M = (2l+1)^{\frac{1}{2l+1}}$, $N = ml_2 M$, $L = \frac{l}{2l+1}$. (38)

We observe that the solutions of five dimensional Bianchi-III space-time are in the format of Pradhan et al (2011).

§4. Some physical and geometric properties of the model

From the field equations (7) –(12) , we obtain

$$8\pi(\lambda + p) = \frac{L(6L-3)}{\beta^2 T^2} - \frac{a^2}{N^2 T^{2L}} + \Lambda\,, \tag{39}$$

$$8\pi p = \frac{L^2[1+2l+3l^2] - Ll(2l+1)}{l^2\beta^2 T^2}) + \Lambda\,, \tag{40}$$

$$8\pi\rho = \frac{-l(3l+3)L^2}{l^2\beta^2 T^2} + \frac{a^2}{N^2 T^{2L}} - \Lambda\,, \tag{42}$$

Now we consider equation of state

$$p = \gamma\rho\,, \tag{43}$$

where $\gamma(0 \le \gamma \le 1)$ is a constant.

The isotropic pressure p, the rest energy density (ρ), the string tension density (λ) and the particle density (ρ_p) for the model (37) are given by

$$8\pi(1+\gamma)p = \frac{\gamma a^2}{N^2 T^{2L}} - \frac{\gamma L\{L(l-1) + l(2l+1)\}}{l^2\beta^2}\frac{1}{T^2}\,, \tag{44}$$

$$8\pi(1+\gamma)\rho = \frac{a^2}{N^2 T^{2L}} - \frac{L\{L(l-1) + l(2l+1)\}}{l^2\beta^2}\frac{1}{T^2}\,, \tag{45}$$

$$8\pi\lambda = \frac{a^2}{N^2 T^{2L}} - \frac{L\{L(3l^2 - 2l - 1] + l(1-l)\}}{l^2\beta^2 T^2}\,, \tag{46}$$

$$8\pi(1+\gamma)\rho_p = \frac{(\gamma+2)a^2}{N^2 T^{2L}} - \frac{L}{l^2\beta^2 T^2}[L\{\gamma(3l^2 - 2l - 1) + 3l^2 - l - 2\} + l[\gamma(1-l) + l + 2] \tag{47}$$

$$(1+\gamma)\Lambda = \frac{\gamma a^2}{N^2 T^{2L}} + \frac{L}{l^2\beta^2 T^2}[l(2l+1) - L(3l^2 + 2l + 1) + \gamma l(3l+3)]. \tag{48}$$

From (45) and (47) we observe that the energy conditions $\rho \ge 0$, $\rho_p \ge 0$ are satisfied under the conditions

$$L \ge \frac{l(2l+1)}{l-1} \ge 0 \qquad \text{or} \qquad 0 \ge \frac{l(2l+1)}{l-1} \ge L\,. \tag{49}$$

and

$$\frac{l[\gamma(l-1)-l-2]}{\gamma[(3l^2-2l-1)+(3l^2-l-2)}\le L\le 0 \quad \text{or} \quad 0\le L\le \frac{l[\gamma(l-1)-l-2]}{\gamma(3l^2-2l-1)+(3l^2-l-2)} \tag{50}$$

We also observe that string tension density $\lambda \ge 0$ and cosmological constant $\Lambda \ge 0$ under conditions

$$T \le \left[\frac{N^2 L\{L(3l^2-2l-1)+l(1-l)\}}{a^2 l^2 \beta^2}\right]^{\frac{1}{2(1-L)}}, \tag{51}$$

and

$$T \ge \left[\frac{N^2 L}{a^2 l^2 \beta^2 \gamma}\{L(3l^2+2l+1)+l(3l+3)\gamma\}-l(2l+1)\right]^{\frac{1}{2(1-L)}} \tag{52}$$

respectively.

All the physical quantities p, ρ, ρ_p, λ and Λ tend to infinity at $T=0$ and 0 at $T=\infty$. Therefore, the model (37) start expanding with big bang singularity with $T=0$ until it comes to rest at $T=\infty$,This is point type singularity because the directional scale parameters $A(t), B(t)$ and $C(t)$ are vanishes at $T=0$. We also note that $T=0$ and $T=\infty$ respectively correspond to the proper time $t=t_0$ and $t=\infty$, where $t_0=-k_2\beta$.

Either for ($L<1$ and $T\to 0$) or for ($L>1$ and $T\to\infty$), we get

$$\frac{\rho_p}{\lambda} = -\frac{[L\{\gamma(3l^2-2l-1)+3l^2-l-2\}+l[\gamma(1-l)+l+2]}{(1+\gamma)\{L[3l^2-2l-1]+l(1-l)}. \tag{53}$$

Also either for $L>1$ and $T\to 0$) or for ($L<1$ and $T\to\infty$), we obtain

$$\frac{\rho_p}{\lambda} = -\frac{\gamma+2}{\gamma+1}. \tag{54}$$

In this case the particle density and the tension density of the string are comparable at the two ends and they fall off asymptotically at similar rate.

According to Refs. (see Kibble 1976; Krori et al. 1990), when $\rho_p/|\lambda| > 1$, in the process of evolution, the universe is dominated by massive strings, and when $\rho_p/|\lambda| < 1$, the strings dominates over the particle.

From (53), we observe

$$\frac{\rho_p}{|\lambda|} > 1 \quad \text{when} \quad L < \frac{l(l-1)(1+2\gamma) - l^2 - 2l}{(1+\gamma)[6l^2 - 2l - 1] - \gamma[2l+1] - 2}.$$

Thus, in these cases, the universe is dominated by massive strings throughout the whole process of evolution of the universe in early stage as well as at late time. Also from (53), we note that when

$$L > \frac{l(l-1)(1+2\gamma) - l^2 - 2l}{(1+\gamma)[6l^2 - 2l - 1] - \gamma[2l+1] - 2} \quad \text{, we get} \quad \frac{\rho_p}{|\lambda|} < 1$$

which implies that strings dominates over the particle.

From (54), we observe $\frac{\rho_p}{|\lambda|} > 1$

Hence, also in these cases, the universe is dominated by massive strings throughout the whole process of evolution of our model in early stage as well as at late time.

The expressions for the scalar of expansion θ, magnitude of shear σ^2, the average anisotropy parameter A_m, deceleration parameter q and proper volume V for the model (37) are given by

$$\theta = \frac{(3l+1)L}{l\beta T}, \tag{55}$$

$$\sigma^2 = \frac{1}{4}\frac{3}{2}\left[\frac{(l-1)L}{l\beta T}\right]^2, \tag{56}$$

$$A_m = 3[\frac{(l-1)}{3l+1}]^2, \tag{57}$$

$$q = -\frac{l\beta}{[3l+1]}, \tag{58}$$

$$V^4 = \frac{N^3}{m} M^{1/l} T^{\frac{L[3l+1]}{l}}. \tag{59}$$

The rate of expansion H_i in the direction of x , y, z, u are given by

$$H_1 = H_2 = H_3 = \frac{L}{\beta T}, \tag{60}$$

$$H_4 = \frac{L}{l\beta T}. \tag{61}$$

Hence the average generalized Hubble's parameter is given by

$$H = \frac{(3l+1)L}{4l\beta T}. \tag{62}$$

Observations : From the above results we observe that

(1) The spatial volume is zero at $T = 0$ and it increases with the increase of T. This shows that the universe starts evolving with zero volume at $T = 0$ and expands with cosmic time T.

(2) From equation (60)-(61), we observe that all the three directional Hubble parameters are zero at $T \to \infty$.

(3) In derived model, the energy density tend to infinity at $T = 0$. The model has the point-type singularity at $T = 0$

(4) The energy density becomes zero as $T \to \infty$.

(5)The shear scalar diverse at $T = 0$.

(6)The expansion scalar and shear scalar all tend to zero as $T \to \infty$.

(7) As $T \to \infty$, the scale factors $A(t)$, $B(t)$ and $C(t)$ tend to infinity.

(8) The cosmological evolution of five dimensional Bianchi type-III space-time is expansionary, with all the three scale factors monotonically increasing function of time

(9)The mean anisotropy parameter are uniform throughout whole expansion of the universe when $l \neq -\frac{1}{3}$ but for $l = -\frac{1}{3}$ it tends to infinity. This shows that the universe is expanding with the increase of cosmic time but the rate of expansion and shear scalar decrease to zero and tend to isotropic.

(10) At the initial stage of expansion, when ρ is large, the Hubble parameter is also large and with the expansion of the universe H, θ decrease as does ρ.

(11) Since $\frac{\sigma^2}{\theta^2} = \text{constant}$ provided $l \neq -\frac{1}{3}$, the model does not approach isotropy at any time.

(12) The dynamics of the mean anisotropy parameter depends on the value of l .

From (58) we observe that (i) for $l < -\frac{1}{3}$, $q > 0$ i.e., the model is decelerating and (ii) for $l > -\frac{1}{3}$, $q < 0$ i.e., the model is accelerating. Thus our model of the universe is consistent with the recent observations.

Now here we give graphical representation of the model

From (45), we note that $\rho(t)$ is a decreasing function of time and $\rho > 0$ for all times as shown in Fig.1

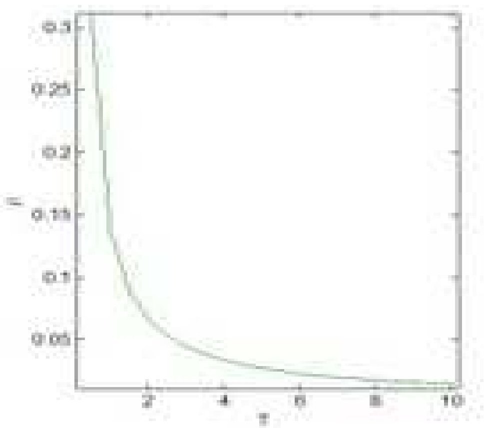

Fig1 Proper density ρ versus time t

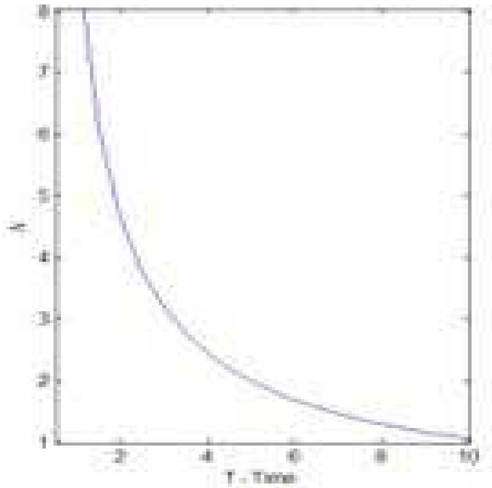

Fig.2 Cosmological constant versus Time

The cosmological termwill be determined the behavior of the universe in this model ; this term has the same effect as a uniform mass density, which is constant in time. A positive value of corresponds to a negative effective mass density (repulsion). Hence, we expect that in the universe with a positive value of, the expansion will tends to accelerate; whereas in the universe with negative value of, the expansion will slow down, stop and reverse. From (48), we see that the cosmological term is a decreasing function of time and it approaches a small positive value at late time. From Fig.2, we note this behavior of cosmological term in the model.

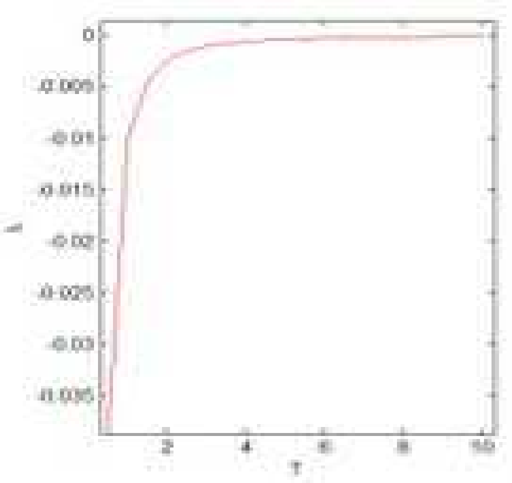

Fig.3 Tension density versus Time

From Fig.3, we see that the string tension density is always negative. It is pointed out by Letelier (1978,1983) that may be positive or negative. If, the string phase of the universe disappears, i.e., we have an anisotropic fluid of particles.

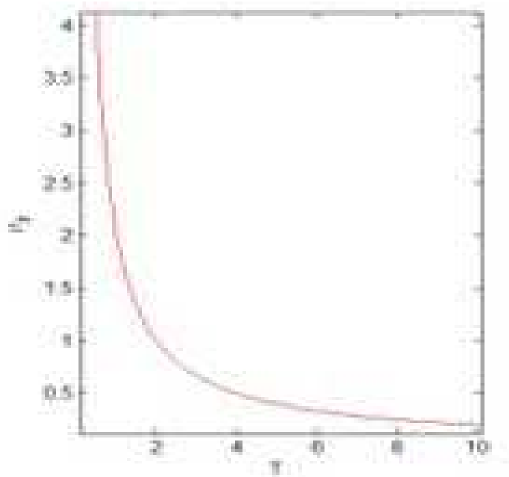

Fig.4 Particle density versus Time

From (47), we note that the particle density is a decreasing function of time and for all times. This nature of is clearly shown in Fig.4.

§ 5. Conclusion :

We conclude that the derived model starts expanding with big bang singularity at until it comes to rest at.This singularity is a point type at initial moment. It is to be noted that and correspond to proper time and**.** We observe that is constant provided, the model does not approach isotropy through the whole evolution of the universe. Our model (37) is in accelerating phase which is consistent to the recent observations. The particle density and the tension density of the string are comparable at the two ends and they fall off asymptotically at similar rate. It is observed that universe in dominated by massive string throughout the whole process of evolution of our model in early stage as well as late time. It is also observed that string dominates over the particle in the early stage of evolution of the universe.

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