Stability of Collinear Libration Point L_1 in Photo – Gravitational Restricted Problem of 2+2 Bodies When Bigger Primary is a Triaxial Rigid Body Perturbed by Coriolis and Centrifugal Forces

Ziaul Hoque[a], D. N. Garain[b]
[a]Department of Mathematics, S. P. College, Dumka, Jharkhand, INDIA
[b]P.G. Department of Mathematics, S. K. M. University, Dumka, Jharkhand, INDIA

Abstract

Stability of collinear libration point L_1 in photo – gravitational restricted problem of 2+2 bodies when bigger primary is a triaxial rigid body perturbed by coriolis and centrifugal forces has been studied in which it is found that L_1 is unstable.

KEYWORDS: Stability, Collinear libration point, Photo-gravitation, Triaxial rigid body, Coriolis force, Centrifugal force.

1. INTRODUCTION

Equilibrium solutions of restricted problem of 2+2 bodies are derived by Whipple (1984) in which M_1 and M_2 are two point masses moving in the circular Keplerian orbit about their centre of mass. He assumed that $M_1 \geq M_2$. Two minor bodies m_1 and m_2 $(m_1, m_2 << M_2)$ move in the gravitational fields of primaries $(M_1 \text{ and } M_2)$. They attract each other but do not perturb the primaries. He showed the existence of fourteen equilibrium solutions. Six of these solutions are located about the collinear Lagrangian points of classical restricted problem of three bodies, eight solutions are found in the neighborhood of triangular Lagrangian points.

Sharma, Taqvi and Bhatnagar (2001) studied the existence and stability of libration points in the restricted three body problem when primaries are triaxial rigid body and source of radiation. They found five libration points, two triangular and three collinear they also observed that collinear points are unstable while triangular libration points are stable for the mass parameter $0 \leq \mu < \mu_{crit}$.

Garain and Chakraborty (2007) derived libration points and examined stability in Robe's three body restricted problem when second primary is a triaxial rigid body perturbed by coriolis and centrifugal forces. They found that the collinear libration points are deviated due to perturbation of centrifugal forces and triaxility of the second primary. Perturbation of coriolis force and triaxility character of the body play important role for finding the region of stability.

Hoque and Garain (2013) computed collinear libration point L_1 in the case of 2+2 body problem when perturbation effects act in coriolis and centrifugal forces, small primary is a radiating body and bigger primary as a triaxial rigid body.

2. EQUATIONS OF MOTION

Whipple's (1984) equation of motion of restricted problem of 2 + 2 bodies in synodic system be

$$\ddot{x}_i - 2\dot{y}_i = \frac{\partial T}{\partial x_i} \tag{1}$$

$$\ddot{y}_i + 2\dot{x}_i = \frac{\partial T}{\partial y_i} \tag{2}$$

$$\ddot{z}_i = \frac{\partial T}{\partial z_i}, \qquad (\text{i} = 1, 2) \tag{3}$$

$$T = \sum_{i=1}^{2} \mu_i \left[\frac{(x_i^2 + y_i^2)}{2} + \frac{(1-\mu)}{r_{1i}} + \frac{\mu}{r_{2i}} + \frac{\mu_{3-i}}{2r} \right]$$

$$\mu = \frac{M_2}{M_1 + M_2}, \mu_i = \frac{m_i}{M_1 + M_2}, \ (\text{i} = 1, 2)$$

$$r_{1i}^2 = (x_i - \mu)^2 + y_i^2 + z_i^2, \quad r_{2i}^2 = (x_i - \mu + 1)^2 + y_i^2 + z_i^2$$

$$r^2 = (x_1 - x_2)^2 + (y_1 - y_{2)})^2 + (z_1 - z_2)^2$$

Here we consider M_2, a radiating body and M_1, a triaxial rigid body, we have also considered effect of perturbation in coriolis and centrifugal forces in this configuration.

Hence force function T reduces to U as follows:

$$U = \sum_{i=1}^{2} \mu_i \left[\frac{\beta(x_i^2 + y_i^2)}{2} + \frac{(1-\mu)}{r_{1i}} + \frac{q\mu}{r_{2i}} + \frac{\mu_{3-i}}{2r} + \frac{(1-\mu)(2\sigma_1 - \sigma_2)}{2r_{1i}^3} - \frac{3(1-\mu)(\sigma_1 - \sigma_2)y_i^2}{2r_{1i}^5} \right] \tag{4}$$

Where, $q = 1 - \in, \beta = 1 + \in'$ and $i = 1, 2$.

Equilibrium points of the system are those points where

$$\dot{x}_i = \dot{y}_i = \dot{z}_i = \frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial y_i} = \frac{\partial U}{\partial z_i} = 0, \ (\text{i} = 1, 2).$$

Thus we get

$$\beta x_1 - \frac{(1-\mu)(x_1 - \mu)}{r_{11}^3} - \frac{q\mu(x_1 - \mu + 1)}{r_{21}^3} - \frac{\mu_2(x_1 - x_2)}{r^3} - \frac{3(1-\mu)(2\sigma_1 - \sigma_2)(x_1 - \mu)}{2r_{11}^5}$$

$$+ \frac{15(1-\mu)(\sigma_1 - \sigma_2)(x_1 - \mu)y_1^2}{2r_{11}^7} = 0 \tag{5}$$

$$\beta y_1 - \frac{(1-\mu)y_1}{r_{11}^3} - \frac{q\mu y_1}{r_{21}^3} - \frac{\mu_2(y_1 - y_2)}{r^3} - \frac{3(1-\mu)(2\sigma_1 - \sigma_2)y_1}{2r_{11}^5}$$

$$- \frac{3(1-\mu)(\sigma_1 - \sigma_2)}{2} \left\{ \frac{2y_1}{r_{11}^5} - \frac{5y_1^3}{r_{11}^7} \right\} = 0 \tag{6}$$

$$- \frac{(1-\mu)z_1}{r_{11}^3} - \frac{q\mu z_1}{r_{21}^3} - \frac{\mu_2(z_1 - z_2)}{r^3} - \frac{3(1-\mu)(2\sigma_1 - \sigma_2)z_1}{2r_{11}^5} + \frac{15(1-\mu)(\sigma_1 - \sigma_2)y_1^2 z_1}{2r_{11}^7} = 0 \tag{7}$$

$$\beta x_2 - \frac{(1-\mu)(x_2 - \mu)}{r_{12}^3} - \frac{q\mu(x_2 - \mu + 1)}{r_{22}^3} - \frac{\mu_1(x_2 - x_1)}{r^3} - \frac{3(1-\mu)(2\sigma_1 - \sigma_2)(x_2 - \mu)}{2r_{12}^5}$$

$$+ \frac{15(1-\mu)(\sigma_1 - \sigma_2)(x_2 - \mu)y_2^2}{2r_{12}^7} = 0 \tag{8}$$

$$\beta y_2 - \frac{(1-\mu)y_2}{r_{12}^3} - \frac{q\mu y_2}{r_{22}^3} - \frac{\mu_1(y_2 - y_1)}{r^3} - \frac{3(1-\mu)(2\sigma_1 - \sigma_2)y_2}{2r_{12}^5}$$

$$-\frac{3(1-\mu)(\sigma_1-\sigma_2)}{2}\left\{\frac{2y_2}{r_{12}^5}-\frac{5y_2^3}{r_{12}^7}\right\}=0 \tag{9}$$

$$-\frac{(1-\mu)z_2}{r_{12}^3}-\frac{q\mu z_2}{r_{22}^3}-\frac{\mu_1(z_2-z_1)}{r^3}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)z_2}{2r_{12}^5}+\frac{15(1-\mu)(\sigma_1-\sigma_2)y_2^2z_2}{2r_{12}^7}=0 \tag{10}$$

From (7) and (10), we get $z_1 = z_2 = 0$.

By inspection, it can be seen that equations (6) and (9) are satisfied when $y_1 = y_2 = 0$. Now we have to determine x_1 and x_2 such that the following simplified forms of equations (5) and (8) are satisfied.

$$\therefore\ \beta x_1-\frac{(1-\mu)(x_1-\mu)}{|x_1-\mu|^3}-\frac{q\mu(x_1-\mu+1)}{|x_1-\mu+1|^3}-\frac{\mu_2(x_1-x_2)}{|x_1-x_2|^3}$$
$$-\frac{3(1-\mu)(2\sigma_1-\sigma_2)(x_1-\mu)}{2|x_1-\mu|^5}+\frac{15(1-\mu)(\sigma_1-\sigma_2)(x_1-\mu)y_1^2}{2|x_1-\mu|^7}=0 \tag{11}$$

And $\beta x_2-\frac{(1-\mu)(x_2-\mu)}{|x_2-\mu|^3}-\frac{q\mu(x_2-\mu+1)}{|x_2-\mu+1|^3}-\frac{\mu_2(x_2-x_1)}{|x_2-x_1|^3}$

$$-\frac{3(1-\mu)(2\sigma_1-\sigma_2)(x_2-\mu)}{2|x_2-\mu|^5}+\frac{15(1-\mu)(\sigma_1-\sigma_2)(x_2-\mu)y_2^2}{2|x_2-\mu|^7}=0 \tag{12}$$

The solution of equations (11) and (12) can be obtained with the help of power series.

$$\text{Let}\in_i=\frac{\mu_i}{(\mu_1+\mu_2)^{\frac{2}{3}}},\ (i=1,2) \tag{13}$$

$$\therefore\in_1=\frac{\mu_1}{(\mu_1+\mu_2)^{\frac{2}{3}}}\ \text{and}\ \in_2=\frac{\mu_2}{(\mu_1+\mu_2)^{\frac{2}{3}}} \tag{14}$$

$$\therefore\mu_2\in_1=\frac{\mu_1\mu_2}{(\mu_1+\mu_2)^{\frac{2}{3}}}=\mu_1\in_2=k(say) \tag{15}$$

$$\text{Let}\ x_1=L_i'+\sum_{j=1}^{n}a_{1j}\in_2^j,\ \text{for}\ i=1,\ 2,\ 3 \tag{16}$$

Where L_1', L_2', L_3' be equilibrium points in photo-gravitational restricted problem of three bodies when bigger primary is a triaxial rigid body perturbed by coriolis and centrifugal forces and x_1 be the x coordinate of first small body.

$$x_2=L_i'+\sum_{j=1}^{n}a_{2j}\in_1^j\ \text{for}\ i=1,2,3 \tag{17}$$

Similar to Whipple,

$$x_1=L_i'+\frac{(\pm1)}{(\Omega_{xx}^{\circ})^{\frac{1}{3}}}\frac{\mu_2}{(\mu_1+\mu_2)^{\frac{2}{3}}}\ \text{where}\ i=1,2,3 \tag{18}$$

And

$$x_2 = L_i' - \frac{(\pm 1)}{(\Omega_{xx}^{\circ})^{\frac{1}{3}}} \frac{\mu_1}{(\mu_1+\mu_2)^{\frac{2}{3}}} \quad \text{where } i = 1,2,3 \tag{19}$$

Hoque and Garain (2013) obtained two values $(x_1, 0, 0)$ and $(x_2, 0, 0)$ of L_1 in which

$x_1 = a_{11} + b_{11} \in + c_{11} \in' -2d_{11}\sigma_1 + d_{11}\sigma_2$

and $x_2 = a_{12} + b_{12} \in + c_{12} \in' -2d_{12}\sigma_1 + d_{12}\sigma_2$

Where , $a_{11} = \mu - 1 - a_1 + \frac{(\pm 1)\mu_2}{A_1^{\frac{1}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$, $\mathrm{b}_{11} = b_1 - \frac{(\pm 1)\mu_2 B_1}{3A_1^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$

$c_{11} = c_1 - \frac{(\pm 1)\mu_2 C_1}{3A_1^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$, $d_{11} = d_1 - \frac{(\pm 1)\mu_2 D_1}{3A_1^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$

$a_{12} = \mu - 1 - a_1 - \frac{(\pm 1)\mu_1}{A_1^{\frac{1}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$, $\mathrm{b}_{12} = b_1 + \frac{(\pm 1)\mu_1 B_1}{3A_1^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$

$c_{12} = c_1 + \frac{(\pm 1)\mu_1 C_1}{3A_1^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$, $d_{12} = d_1 + \frac{(\pm 1)\mu_1 D_1}{3A_1^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$

They obtained the particular values of x_1 and x_2 for different values of μ, μ_1 and μ_2. Our characteristic equation corresponding to the point $(x_1, 0, 0)$ is

$$f(\lambda) = \lambda^4 + \lambda^2\left(4 - \frac{U_{x_1x_1}}{\mu_1} - U_{y_1y_1}\right) + \frac{1}{\mu_1^2}\left(U_{x_1x_1}U_{y_1y_1} - U_{x_1y_1}^2\right) = 0 \tag{20}$$

$$\Rightarrow \frac{U_{x_1x_1}}{\mu_1} = \left[\beta + \frac{2(1-\mu)}{r_{11}^3} + \frac{2q\mu}{r_{21}^3} + \frac{2\mu_2}{r^3} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{r_{11}^5} - \frac{45(1-\mu)(\sigma_1-\sigma_2)y_1^2}{r_{11}^7}\right]$$

$$\Rightarrow \frac{U_{y_1y_1}}{\mu_1} = \left[\beta - \frac{(1-\mu)}{r_{11}^3} - \frac{q\mu}{r_{21}^3} - \frac{\mu_2}{r^3} - \frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2r_{11}^5} + \frac{3(1-\mu)y_1^2}{r_{11}^5} + \frac{3q\mu y_1^2}{r_{21}^5}\right.$$

$$\left. + \frac{3\mu_2(y_1-y_2)^2}{r^5} + \frac{15(1-\mu)(7\sigma_1-6\sigma_2)y_1^2}{2r_{11}^7} - \frac{105(1-\mu)(\sigma_1-\sigma_2)y_1^4}{2r_{11}^9}\right]$$

$$U_{x_1y_1} = \mu_1\left[\frac{3(1-\mu)(x_1-\mu)y_1}{r_{11}^5} + \frac{3q\mu(x_1-\mu+1)y_1}{r_{21}^5} + \frac{3\mu_2(x_1-x_2)(y_1-y_2)}{r^5}\right.$$

$$\left. + \frac{15(1-\mu)(2\sigma_1-\sigma_2)(x_1-\mu)y_1}{2r_{11}^7} + \frac{15(1-\mu)(\sigma_1-\sigma_2)(x_1-\mu)y_1}{r_{11}^7}\right.$$

In this case, $y_1 = 0$ and $y_2 = 0$.

∴The above equations reduce to

$$\frac{U_{x_1x_1}}{\mu_1} = \left[\beta + \frac{2(1-\mu)}{(x_1-\mu)^3} + \frac{2q\mu}{(x_1-\mu+1)^3} + \frac{2\mu_2}{(x_1-x_2)^3} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{(x_1-\mu)^5}\right]$$

$$\frac{U_{y_1y_1}}{\mu_1}=\left[\beta-\frac{(1-\mu)}{(x_1-\mu)^3}-\frac{q\mu}{(x_1-\mu+1)^3}-\frac{\mu_2}{(x_1-x_2)^3}-\frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2(x_1-\mu)^5}\right]$$

And $\frac{U_{x_1y_1}}{\mu_1}=0$

For the collinear equilibrium solutions the partial derivatives contained in equation (20) reduce to

$$\frac{U_{x_1x_1}}{\mu_1}=\left[\beta+\frac{2(1-\mu)}{|x_1-\mu|^3}+\frac{2q\mu}{|x_1-\mu+1|^3}+\frac{2\mu_2}{|x_1-x_2|^3}+\frac{6(1-\mu)(2\sigma_1-\sigma_2)}{|x_1-\mu|^5}\right]$$

$$\frac{U_{x_1y_1}}{\mu_1}=0$$

$$\frac{U_{y_1y_1}}{\mu_1}=\left[(1+\in')-\frac{(1-\mu)}{|x_1-\mu|^3}-\frac{(1-\in)\mu}{|x_1-\mu+1|^3}-\frac{\mu_2}{|x_1-x_2|^3}-\frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2|x_1-\mu|^5}\right]$$

Let $\frac{U_{y_1y_1}}{\mu_1}=A_{11}+B_{11}\in+C_{11}\in'-D_{11}\sigma_1+E_{11}\sigma_2$

$x_1=a_{11}+b_{11}\in+c_{11}\in'-2d_{11}\sigma_1+d_{11}\sigma_2$

And

$x_2=a_{12}+b_{12}\in+c_{12}\in'-2d_{12}\sigma_1+d_{12}\sigma_2$

$\Rightarrow\frac{U_{y_1y_1}}{\mu_1}=A_{11}+B_{11}\in+C_{11}\in'-D_{11}\sigma_1+E_{11}\sigma_2$

Where, $A_{11}=1-\frac{(1-\mu)}{|a_{11}-\mu|^3}-\frac{\mu}{|a_{11}-\mu+1|^3}-\frac{\mu_2}{|a_{11}-a_{12}|^3}$,

$B_{11}=\frac{3(1-\mu)b_{11}}{|a_{11}-\mu|^4}+\frac{3\mu b_{11}}{|a_{11}-\mu+1|^4}+\frac{\mu}{|a_{11}-\mu+1|^3}+\frac{3\mu_2(b_{11}-b_{12})}{|a_{11}-a_{12}|^4}$,

$C_{11}=1+\frac{3(1-\mu)c_{11}}{|a_{11}-\mu|^4}+\frac{3\mu c_{11}}{|a_{11}-\mu+1|^4}+\frac{3\mu_2(c_{11}-c_{12})}{|a_{11}-a_{12}|^4}$,

$D_{11}=\frac{6(1-\mu)d_{11}}{|a_{11}-\mu|^4}+\frac{6\mu d_{11}}{|a_{11}-\mu+1|^4}+\frac{6\mu_2(d_{11}-d_{12})}{|a_{11}-a_{12}|^4}+\frac{6(1-\mu)}{|a_{11}-\mu|^5}$

And $E_{11}=\frac{3(1-\mu)d_{11}}{|a_{11}-\mu|^4}+\frac{3\mu d_{11}}{|a_{11}-\mu+1|^4}+\frac{3\mu_2(d_{11}-d_{12})}{|a_{11}-a_{12}|^4}+\frac{9(1-\mu)}{2|a_{11}-\mu|^5}$

Here we see that $U_{x_1x_1}>0$ and $U_{x_1y_1}=0$. Equation (6.2.1) reduces to

$$f(\lambda)=\lambda^4+\lambda^2\left(4-\frac{1}{\mu_1}U_{x_1x_1}-U_{y_1y_1}\right)+\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}=0 \qquad (20a)$$

Case I: Let λ_1^2 and λ_2^2 be the roots of equation (20a)

Sub case (i) if λ_1^2 and λ_2^2 both are real and one of them is positive, let λ_1^2 is a positive quantity, then square root of λ_1^2 must be real and of opposite sign. In this case

characteristic roots will be a real number. So in this case L_1 is unstable. We may get similar result in the case of λ_2^2.

Sub case (ii) if λ_1^2 and λ_2^2 both are real and negative, let λ_1^2 is a negative quantity, then two roots of λ_1^2 be purely imaginary. Similarly two roots of λ_2^2 be purely imaginary. In this case L_1 should be stable. Again when λ_1^2 and λ_2^2 both are negative quantity then

$\lambda_1^2\lambda_2^2$ = a positive quantity (20b)

From equation (6.2.1a), we have obtained that

$$\lambda_1^2\lambda_2^2 = \frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} = \text{a negative quantity } \left(\because U_{x_1x_1} > 0,\ U_{y_1y_1} < 0\right) \quad (20c)$$

Since (20b) and (20c) contradict each other. So L_1 is unstable.

Now let $4 - \frac{U_{x_1x_1}}{\mu_1} - U_{y_1y_1} = A_{13}$

Case II: If $A_{13} > 0$ i.e., $4 > \frac{1}{\mu_1}U_{x_1x_1} + U_{y_1y_1}$ then $f(\lambda) = \lambda^4 + A_{13}\lambda^2 + \frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} = 0$

.

Sub case (i) it is clear that $U_{x_1x_1} > 0$ and let $U_{y_1y_1} > 0$ then $\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} > 0$ so, $f(\lambda) = 0$ has no change of sign and as such it has no positive real root. $f(-\lambda) = \lambda^4 + A_{13}\lambda^2 + \frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} = 0$ also has no change of sign and as such it has no positive real root i.e., $f(\lambda)$ has no negative real root.

So in this case we can say that all roots of $f(\lambda) = 0$ are imaginary.

Sub case (ii) if $U_{y_1y_1} < 0$ then similar to Yadav's case $\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} < 0$.

Let, $\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} = -B_{13}, B_{13} > 0$. So, $f(\lambda) = \lambda^4 + A_{13}\lambda^2 - B_{13} = 0$. Here $f(\lambda) = 0$ has only one change of sign and as such it has one positive real root. $f(-\lambda) = \lambda^4 + A_{13}\lambda^2 - B_{13} = 0$ has only one change of sign and as such it has one positive real root i.e., $f(\lambda)$ has one negative real root.

Case III: If $A_{13} < 0$ i.e., $4 < \frac{1}{\mu_1}U_{x_1x_1} + U_{y_1y_1}$ then let $A_{13} = -C_{13}, C_{13} > 0$ then $f(\lambda) = \lambda^4 - C_{13}\lambda^2 + \frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} = 0$.

Sub case (i) let $U_{y_1y_1} > 0$, then $\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} > 0$.

Hence, $f(\lambda) = 0$ has two changes of signs and as such it has at most two positive real roots. $f(-\lambda) = \lambda^4 - C_{13}\lambda^2 + \frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} = 0$ also has two change of signs and as such it has at most two positive real roots i.e., $f(\lambda)$ has at most two negative real roots.

Sub case (ii) if $U_{y_1y_1} < 0$, then $\frac{1}{\mu_1^2} U_{x_1x_1} U_{y_1y_1} < 0$.

Let, $\frac{1}{\mu_1^2} U_{x_1x_1} U_{y_1y_1} = -D_{13}, D_{13} > 0$. Here $f(\lambda) = \lambda^4 - C_{13}\lambda^2 - D_{13} = 0$ has only one change of sign and as such it has one positive real root. $f(-\lambda) = \lambda^4 - C_{13}\lambda^2 - D_{13} = 0$ has only one change of sign and as such it has one positive real root i.e., $f(\lambda)$ has one negative real root.

In both the cases we see that $f(\lambda) = 0$ has one positive and one negative real root. So the libration point L_1 is unstable.

The characteristic equation corresponding to the point $(x_2, 0, 0)$ be

$$f(\lambda) = \lambda^4 + \lambda^2\left(4 - \frac{1}{\mu_2} U_{x_2x_2} - U_{y_2y_2}\right) + \frac{1}{\mu_2^2}\left(U_{x_2x_2} U_{y_2y_2} - U_{x_2y_2}^2\right) = 0 \qquad (21)$$

$$\frac{U_{x_2x_2}}{\mu_2} = \left[\beta + \frac{2(1-\mu)}{(x_2-\mu)^3} + \frac{2q\mu}{(x_2-\mu+1)^3} + \frac{2\mu_1}{(x_1-x_2)^3} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{(x_2-\mu)^5}\right]$$

$$\frac{U_{y_2y_2}}{\mu_2} = \left[\beta - \frac{(1-\mu)}{(x_2-\mu)^3} - \frac{q\mu}{(x_2-\mu+1)^3} - \frac{\mu_1}{(x_1-x_2)^3} - \frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2(x_2-\mu)^5}\right]$$

$$\frac{U_{x_2y_2}}{\mu_2} = 0$$

$$\frac{U_{x_2x_2}}{\mu_2} = \left[\beta + \frac{2(1-\mu)}{|x_2-\mu|^3} + \frac{2q\mu}{|x_2-\mu+1|^3} + \frac{2\mu_1}{|x_1-x_2|^3} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{|x_2-\mu|^5}\right]$$

$$\frac{U_{x_2y_2}}{\mu_2} = 0$$

$$\frac{U_{x_2y_2}}{\mu_2} = \left[(1+\epsilon') - \frac{(1-\mu)}{|x_2-\mu|^3} - \frac{(1-\epsilon)\mu}{|x_2-\mu+1|^3} - \frac{\mu_1}{|x_1-x_2|^3} - \frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2|x_2-\mu|^5}\right]$$

Let $\frac{U_{y_2y_2}}{\mu_2} = A_{12} + B_{12}\epsilon + C_{12}\epsilon' - D_{12}\sigma_1 + E_{12}\sigma_2$

$x_1 = a_{11} + b_{11}\epsilon + c_{11}\epsilon' - 2d_{11}\sigma_1 + d_{11}\sigma_2$

And

$x_2 = a_{12} + b_{12}\epsilon + c_{12}\epsilon' - 2d_{12}\sigma_1 + d_{12}\sigma_2$

Where, $A_{12} = 1 - \frac{(1-\mu)}{|a_{12}-\mu|^3} - \frac{\mu}{|a_{12}-\mu+1|^3} - \frac{\mu_1}{|a_{12}-a_{11}|^3}$,

$B_{12} = \frac{3(1-\mu)b_{12}}{|a_{12}-\mu|^4} + \frac{3\mu b_{12}}{|a_{12}-\mu+1|^4} + \frac{\mu}{|a_{12}-\mu+1|^3} + \frac{3\mu_1(b_{12}-b_{11})}{|a_{12}-a_{11}|^4}$,

$$C_{12} = 1 + \frac{3(1-\mu)c_{12}}{|a_{12}-\mu|^4} + \frac{3\mu c_{12}}{|a_{12}-\mu+1|^4} + \frac{3\mu_1(c_{12}-c_{11})}{|a_{12}-a_{11}|^4},$$

$$D_{12} = \frac{6(1-\mu)d_{12}}{|a_{12}-\mu|^4} + \frac{6\mu d_{12}}{|a_{12}-\mu+1|^4} + \frac{6\mu_1(d_{12}-d_{11})}{|a_{12}-a_{11}|^4} + \frac{6(1-\mu)}{|a_{12}-\mu|^5}$$

And $$E_{12} = \frac{3(1-\mu)d_{12}}{|a_{12}-\mu|^4} + \frac{3\mu d_{12}}{|a_{12}-\mu+1|^4} + \frac{3\mu_1(d_{12}-d_{11})}{|a_{12}-a_{11}|^4} + \frac{9(1-\mu)}{2|a_{12}-\mu|^5}$$

Here we see that $U_{x_2x_2} > 0$ and $U_{x_2y_2} = 0$

Equation (6.2.2) reduces to

$$f(\lambda) = \lambda^4 + \lambda^2\left(4 - \frac{1}{\mu_2}U_{x_2x_2} - U_{y_2y_2}\right) + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0 \qquad (21a)$$

Case I: Let λ_1^2 and λ_2^2 be the roots of equation (21a).

Sub case (i) if λ_1^2 and λ_2^2 both are real and one of them is positive, let λ_1^2 is a positive quantity, then square root of λ_1^2 must be real and of opposite sign. In this case characteristic roots will be a real number. So in this case L_1 is unstable. We may get similar result in the case of λ_2^2.

Sub case (ii) if λ_1^2 and λ_2^2 both are real and negative, let λ_1^2 is a negative quantity, then two roots of λ_1^2 be purely imaginary. Similarly two roots of λ_2^2 be purely imaginary. In this case L_1 should be stable. Again when λ_1^2 and λ_2^2 both are negative quantity then

$\lambda_1^2\lambda_2^2$ = a positive quantity (21b)

From equation (21a), we have obtained that

$$\lambda_1^2\lambda_2^2 = \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = \text{a negative quantity } \left(\because U_{x_2x_2} > 0,\ \ U_{y_2y_2} < 0\right) \qquad (21c)$$

Since (21b) and (21c) contradict each other. So L_1 is unstable.

Now let $4 - \frac{1}{\mu_2}U_{x_2x_2} - U_{y_2y_2} = A_{14}$.

Case II: If $A_{14} > 0$ i.e., $4 > \frac{1}{\mu_2}U_{x_2x_2} + U_{y_2y_2}$..

Sub case (i) let $U_{y_2y_2} > 0$ then $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} > 0$ so,

$f(\lambda) = \lambda^4 + A_{14}\lambda^2 + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0$ has no change of sign and as such it has no positive real root. $f(-\lambda) = \lambda^4 + A_{14}\lambda^2 + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0$ also has no change of sign and as such it has no positive real root i.e., $f(\lambda)$ has no negative real root.

So in this case we can say that all roots of $f(\lambda) = 0$ are imaginary.

Sub case (ii) if $U_{y_2y_2} < 0$ then similar to Yadav's case $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} < 0$.

Let, $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = -B_{14}, B_{14} > 0$, so $f(\lambda) = \lambda^4 + A_{14}\lambda^2 - B_{14} = 0$. Here $f(\lambda) = 0$ has only one change of sign and as such it has one positive real root. $f(-\lambda) = \lambda^4 + A_{14}\lambda^2 - B_{14} = 0$ has only one change of sign and as such it has one positive real root i.e., $f(\lambda)$ has one negative real root.

Case III: If $A_{14} < 0$ i.e., $4 < \frac{1}{\mu_2}U_{x_2x_2} + U_{y_2y_2}$ then let $A_{14} = -C_{14}, C_{14} > 0$ then

$$f(\lambda) = \lambda^4 - C_{14}\lambda^2 + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0$$

Sub case (i) let $U_{y_2y_2} > 0$, then $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} > 0$. Then, $f(\lambda) = 0$ has two changes of signs and as such it has at most two positive real roots. $f(-\lambda) = \lambda^4 - C_{14}\lambda^2 + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0$ also has two change of signs and as such it has at most two positive real roots i.e., $f(\lambda)$ has at most two negative real roots.

Sub case (ii) if $U_{y_2y_2} < 0$, then $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} < 0$.

Let, $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = -D_{14}$, $D_{14} > 0$. Here $f(\lambda) = \lambda^4 - C_{14}\lambda^2 - D_{14} = 0$ has only one change of sign and as such it has one positive real root. $f(-\lambda) = \lambda^4 - C_{14}\lambda^2 - D_{14} = 0$ has only one change of sign and as such it has one positive real root i.e., $f(\lambda)$ has one negative real root.

In both cases we see that $f(\lambda) = 0$ has one positive and one negative real root.

3. CONCLUSION

Hence, we get that the libration point L_1 is unstable.

Table 1

For Stability of Collinear Equilibrium Solutions L_1

$$\left(\mu_1 = \mu_2 = 10^{-10}, \frac{1}{\mu_1} U_{y_1 y_1} = A_{11} + B_{11} \in + C_{11} \in' - D_{11}\sigma_1 + E_{11}\sigma_2\right)$$

μ	A_{11}	B_{11}	C_{11}	D_{11}	E_{11}
0.00001	-7.4124765030782400	6.1145040086554300	2.7148638092394600	17.3396244259473000	10.0630996577172000
0.00002	-7.3513309693317400	6.0513657525705100	1.8912271018142300	14.7372962378938000	8.7353055363654700
0.00003	-7.3128315529229700	6.0166482559281200	1.5574854687917000	13.6256735967613000	8.1611413253197100
0.00004	-7.2838982299471000	5.9926042313449700	1.3687333249126700	12.9651298939448000	7.8164488285921700
0.00005	-7.2603808282858300	5.9741528654303400	1.2445796121551000	12.5099565575029000	7.5768141362852100
0.00006	-7.2403942844943100	5.9591427564032700	1.1554521329359300	12.1685278139766000	7.3956646812105300
0.00007	-7.2229123609516100	5.9464642036527500	1.0876949324314400	11.8979571017540000	7.2511235034906800
0.00008	-7.2073102388876900	5.9354705807173500	1.0340528298236800	11.6751525031266000	7.1313711583102900
0.00009	-7.1931772300936600	5.9257522645946100	0.9902825874891190	11.4864283799913000	7.0293800725107300
0.0001	-7.1802278766852100	5.9170334379902800	0.9537218393846350	11.3230845150613000	6.9406688445651200
0.0002	-7.0870671871709300	5.8585957487503000	0.7628409119590420	10.3396672548565000	6.3972656659970400
0.0003	-7.0253801561088100	5.8231012247137700	0.6813597268434320	9.8043947078467300	6.0947223015657800
0.0004	-6.9780029242444300	5.7971434582636600	0.6330793941719900	9.4282928775538400	5.8796536197237100
0.0005	-6.9390510642939900	5.7765153914063900	0.5999165043065280	9.1342457352497800	5.7103332431431500
0.0006	-6.9057315912083800	5.7593235913538100	0.5751178307131290	8.8905595584738000	5.5693805319274500
0.0007	-6.8764782090131100	5.7445455103931600	0.5555214388366990	8.6811138230364500	5.4478641751621400
0.0008	-6.8503163683673500	5.7315630480576200	0.5394253534110580	8.4965709843782500	5.3405686031435500
0.0009	-6.8265947356186900	5.7199725956817100	0.5258216026397380	8.3310243520210900	5.2441725955296800
0.001	-6.8048545479703800	5.7094953150203900	0.5140700242926750	8.1804914668511000	5.1564244794345000
0.002	-6.6491925877964000	5.6381998290989100	0.4436960326870060	7.1187466605272500	4.5364562946454100
0.003	-6.5477215237649200	5.5950151670771800	0.4054890508150690	6.4268840704563300	4.1328851533129100
0.004	-6.4710586071606500	5.5640811924160800	0.3785498039886800	5.8981374231887000	3.8252715240340000
0.005	-6.4089856413995300	5.5401335425359200	0.3573766470484280	5.4643009915697900	3.5735578829883800
0.006	-6.3566324681386600	5.5207343751122600	0.3397301243649620	5.0935399538415500	3.3589877705989700
0.007	-6.3112662804825100	5.5045445469299200	0.3244789694147150	4.7681677257336200	3.1711261995566800
0.008	-6.2711885188116800	5.4907458174651200	0.3109710979171400	4.4772441002500700	3.0035151631811500
0.009	-6.2352661468839400	5.4788001682100400	0.2987954065783780	4.2134860227678900	2.8518552671681700
0.01	-6.2027030429876000	5.4683342615447100	0.2876753335423570	3.9717778920239500	2.7131273198697000
0.02	-5.9804136795764900	5.4077899712556200	0.2080321356155160	2.2425651656375800	1.7277031447253600
0.03	-5.8471506169321100	5.3836249014516500	0.1553920032879180	1.1204307627542000	1.0945563147897800
0.04	-5.7538201941892000	5.3749584616223700	0.1152040416341100	0.2797860248237480	0.6231445134884670
0.05	-5.6838135406961200	5.3751004242349300	0.0825536103626305	-0.3929118635654940	0.2474725278868070
0.06	-5.6293974796681700	5.3810705933620800	0.0550938706867807	-0.9520136494260690	-0.0638719859614674
0.07	-5.5863262627653400	5.3913000684654200	0.0314934107183973	-1.4281665030283200	-0.3285155464982720
0.08	-5.5520180770405300	5.4048756964011900	0.0109127986727788	-1.8405174920936600	-0.5574230956294330
0.09	-5.5247904407006600	5.4212309462185800	-0.0072166287015243	-2.2019067309851500	-0.7579193591067860
0.1	-5.5034917510985400	5.4400001829998800	-0.0233007469211586	-2.5214048303441800	-0.9351605497895840
0.2	-5.5100111757255300	5.7346527213868600	-0.1146694996545430	-4.3519700885728700	-1.9588506566319100
0.3	-5.8393933853015500	6.2461050106455400	-0.1346006989066970	-4.8649964735520600	-2.2714613589495500
0.4	-6.6069793489828200	7.1379465862453300	-0.1113542695003070	-4.6485507794033900	-2.1979305874402400
0.5	-8.2937573222863400	8.9043679419700900	-0.0527563554640675	-3.8928499326952500	-1.8419034681674200

Table 2

For Stability of Collinear Equilibrium Solutions L_1

$$\left(\mu_1 = \mu_2 = 10^{-10}, \frac{1}{\mu_2} U_{y_2 y_2} = A_{12} + B_{12} \in + C_{12} \in' - D_{12}\sigma_1 + E_{12}\sigma_2 \right)$$

μ	A_{12}	B_{12}	C_{12}	D_{12}	E_{12}
0.00001	-7.2443226868681300	5.7798745511389200	140.0024722501930000	385.8665607017090000	194.3246269017590000
0.00002	-7.2181153231065900	5.8119497493043600	113.1801461940610000	305.3023837758250000	154.0159496826790000
0.00003	-7.1965987916724200	5.8262116588903300	100.0786535640900000	265.9478448343970000	134.3203557236860000
0.00004	-7.1783918646589900	5.8344967831639600	91.7824724052607000	241.0262587051750000	121.8451643063780000
0.00005	-7.1625116965874500	5.8399321685933800	85.8656153736733000	223.2512779304240000	112.9456445030720000
0.00006	-7.1483552863574900	5.8437550493943300	81.3432791474586000	209.6650260873180000	106.1420996026970000
0.00007	-7.1355323901856200	5.8465659989922500	77.7259619631768000	198.7972474667870000	100.6989687455150000
0.00008	-7.1237760999724500	5.8486959895217600	74.7377795526352000	189.8192886277510000	96.2016521498576000
0.00009	-7.1128953874169800	5.8503440529283700	72.2092116508317000	182.2219658648770000	92.3953734958735000
0.0001	-7.1027484509998800	5.8516377886590300	70.0292836950863000	175.6719427427970000	89.1133334768473000
0.0002	-7.0257475782151200	5.8560363942639300	57.4396902695300000	137.8386669911610000	70.1450804647468000
0.0003	-6.9719075608867400	5.8554053270659000	51.3159144525614000	119.4311252108960000	60.9064559250616000
0.0004	-6.9294820502343000	5.8535542925196100	47.4490957597011000	107.8052383263860000	55.0665359729681000
0.0005	-6.8940529781462600	5.8513175716701700	44.6974662515116000	99.5306081660422000	50.9069580625893000
0.0006	-6.8634207418512600	5.8489670745435800	42.5984182523365000	93.2172342015096000	47.7311905222694000
0.0007	-6.8363133234726500	5.8466087351786800	40.9223363795786000	88.1751403166313000	45.1933755906422000
0.0008	-6.8119217365838800	5.8442872644615700	39.5399576430892000	84.0158764203397000	43.0987422687965000
0.0009	-6.7896962318478500	5.8420215910384500	38.3719324393067000	80.5009737387212000	41.3276888621613000
0.001	-6.7692444654270900	5.8398188567793400	37.3663598408150000	77.4744461920467000	39.8019622890648000
0.002	-6.6209942096334200	5.8209208277324800	31.5922839602129000	60.0831035135636000	31.0173291798775000
0.003	-6.5231066173678500	5.8058780286900400	28.8137300658685000	51.7009133590012000	26.7686802382493000
0.004	-6.4486971877371700	5.7930538037902400	27.0753347530104000	46.4481424793502000	24.0991185917968000
0.005	-6.3882226593049300	5.7816548181256000	25.8487614492265000	42.7355858140508000	22.2080961471327000
0.006	-6.3370855781151200	5.7712485347339700	24.9206007788247000	39.9211686690823000	20.7717408341143000
0.007	-6.2926881986335700	5.7615731664064400	24.1852079103680000	37.6869817076847000	19.6295087086156000
0.008	-6.2534078548574700	5.7524578747444600	23.5832527785631000	35.8544587098686000	18.6911302559536000
0.009	-6.2181580238805700	5.7437847843916900	23.0783965010535000	34.3142152941402000	17.9012564143520000
0.01	-6.1861729636436700	5.7354690161256400	22.6469208739433000	32.9948484166562000	17.2237249681375000
0.02	-5.9671828884727200	5.6629858355202800	20.2510770698742000	25.5748450906997000	13.3930782400223000
0.03	-5.8354929458318100	5.5987310029708600	19.1801692940671000	22.1349525842154000	11.6011534425972000
0.04	-5.7431407449152800	5.5371061086152200	18.5599882882925000	20.0416956746135000	10.5035061992996000
0.05	-5.6738209777099700	5.4762411904282300	18.1582802768273000	18.5952857806470000	9.7410319733092800
0.06	-5.6199224593942100	5.4152780784081200	17.8826450003589000	17.5172182228032000	9.1702476311590200
0.07	-5.5772595722743100	5.3537439057209400	17.6880205710928000	16.6715448120842000	8.7208794784390900
0.08	-5.5432841413189200	5.2913410046260500	17.5494913087546000	15.9829492814319000	8.3538799748586400
0.09	-5.5163342106173300	5.2278601585465200	17.4520528979710000	15.4059144243356000	8.0455871423370300
0.1	-5.4952715241445900	5.1631394906680300	17.3860976501772000	14.9111228369347000	7.7807222546753500
0.2	-5.5030355824644000	4.4202000159190100	17.6566688091884000	11.9484188859751000	6.1911028149113400
0.3	-5.8327908457930800	3.3889670595585600	19.0318154277233000	10.1450988282707000	5.2334160184873700
0.4	-6.6002203974629100	1.7104036764299100	21.9239659976863000	8.6693853977812000	4.4609109898282700
0.5	-8.2860526381831900	-1.8462597791130300	28.4121273238954000	7.6460116108855000	3.9274297567332800

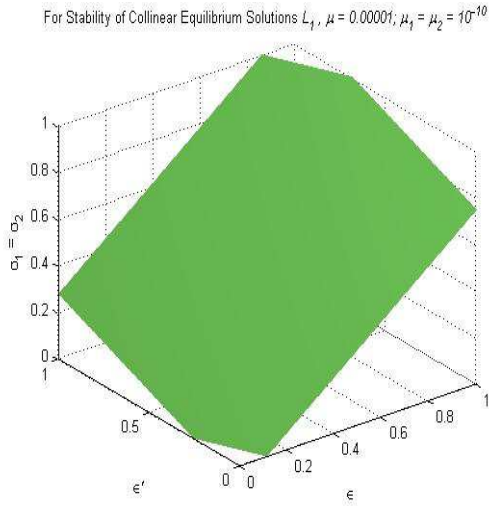

Figure 1

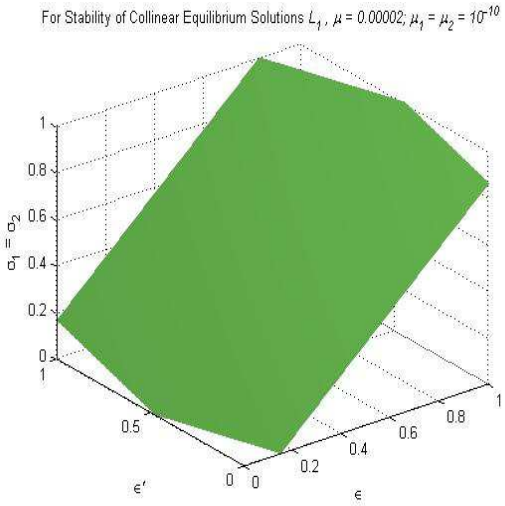

Figure 2

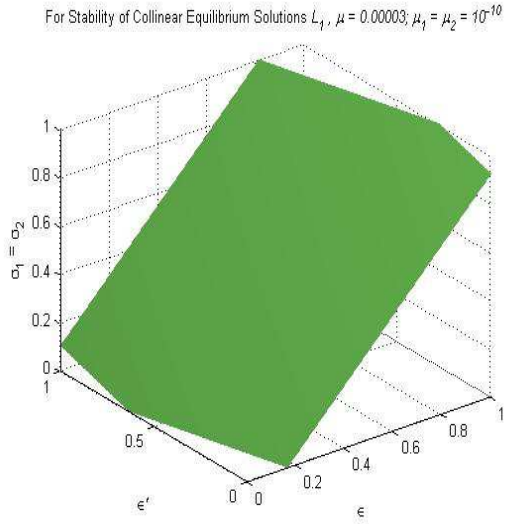

Figure 3

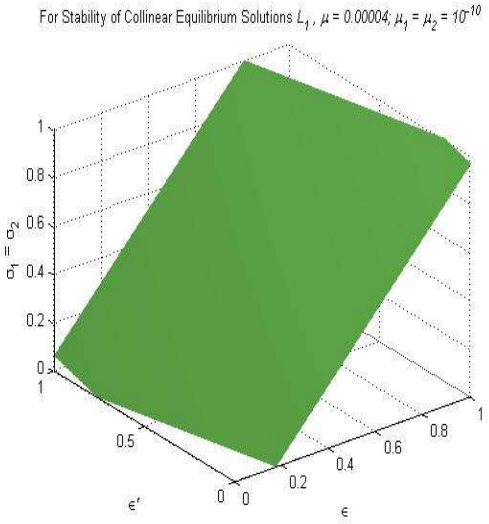

Figure 4

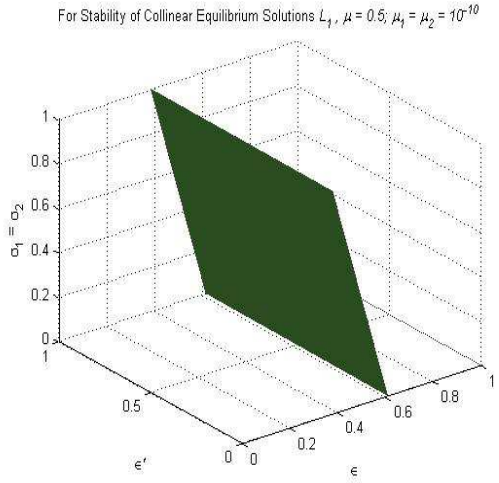

Figure 5

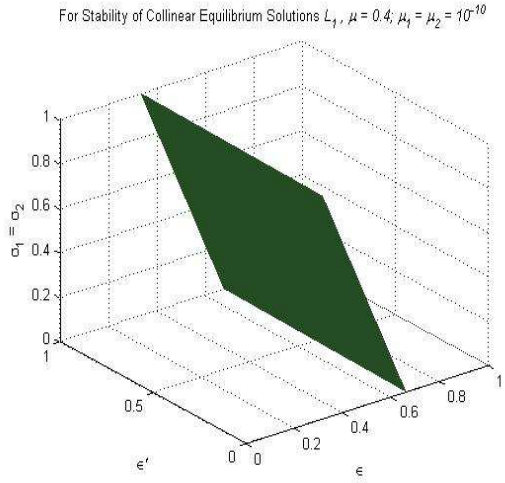

Figure 6

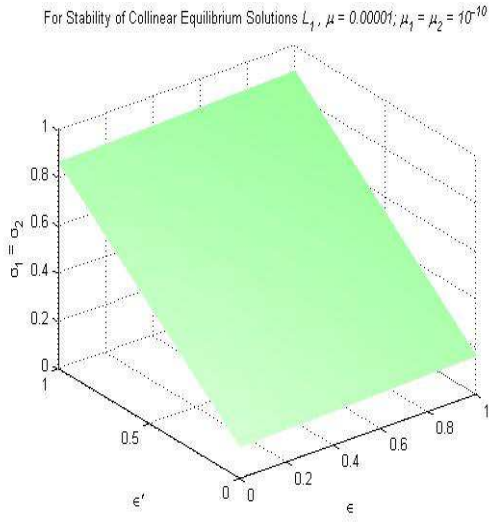

Figure 7

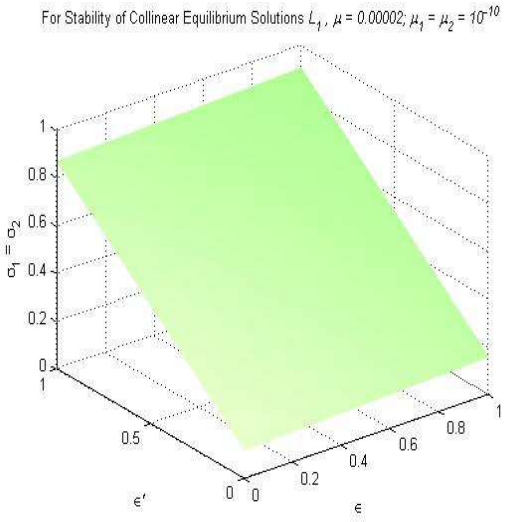

Figure 8

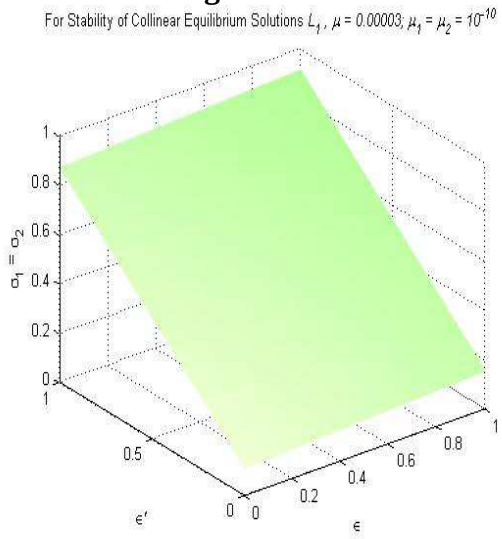

Figure 9

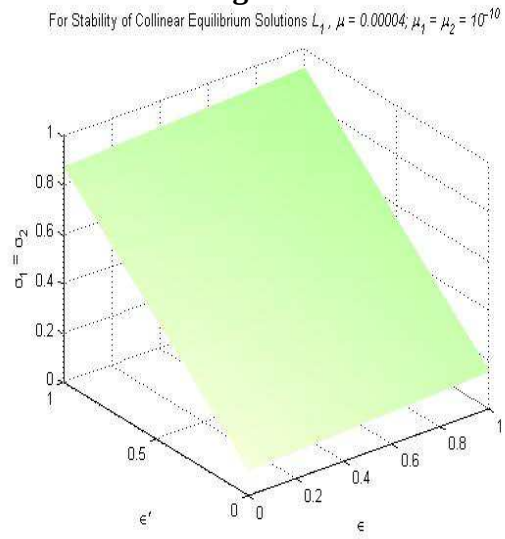

Figure 10

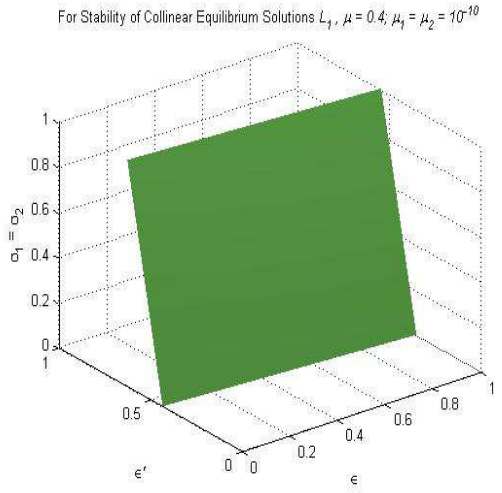

Figure 11

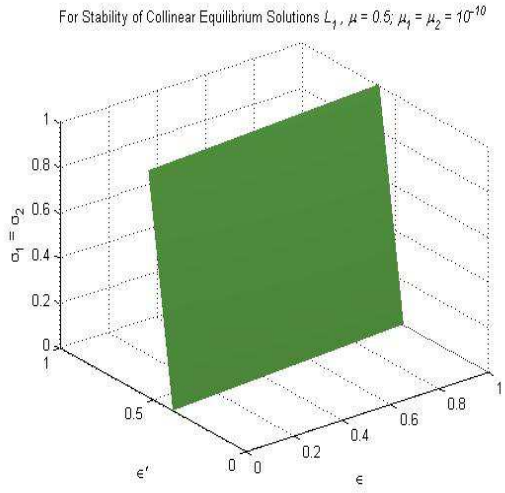

Figure 12

REFERENCES

1. Hoque, Z. and Garain, D. N. (2013): IAJS, Vol. 5, March 2014, pp. 17-22.
2. Sharma, R. K., Taqvi, Z. A. and Bhatnagar, K. B. (2001): *Celestial Mechanics and Dynamical Astronomy, 79,119-133.*
3. Garain, D. N. and Chakraborty, R. (2007): *PROC.NAT.ACAD. SCI. INDIA*, 77(A), IV, 329-332.
4. Whipple, A. L. (1984): Equilibrium solutions of the restricted problem of 2 + 2 bodies, *Celest. Mech*. 33, 271-294.