Vertex and Adjacent Vertex Sum Polynomial of Some Standard Graphs

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Abstract

The vertex polynomial of the graph $G = (V, E)$ is defined by $V(G, x) = \sum_{k=0}^{\Delta(G)} v_k x^k$, where $\Delta(G) = \max\{d(v)/v\epsilon V\}$ and v_k is the number of vertices of degree k. The adjacent vertex sum polynomial of G is defined by $S(G, x) = \sum_{i=o}^{\Delta(G)} n_{\Delta(G)-i} x^{\alpha_{\Delta(G)-i}}$, where $n_{\Delta(G)-i}$ is the sum of the number of adjacent vertices of all the vertices of degree $\Delta(G) - i$ and $\alpha_{\Delta(G)-i}$ is the sum of the degree of adjacent vertices of all the vertices of degree $\Delta(G) - i$. In this paper we seek to find the Vertex Polynomial of Lollipop Graph and Tadpole Graph; Adjacent Vertex Sum Vertex Polynomial of some Cycle related Graphs.

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KEYWORDS: Vertex Polynomial, Adjacent Vertex Sum Polynomial, Splitting graph, Degree splitting graph, Cycle, Lollipop Graph and Tadpole Graph.

1. Introduction:

Here I consider only simple undirected graphs. The terms not defined here we can refer Frank Harary[3]. The vertex set is denoted by V and the edge set denoted by E. For $v\epsilon V$, $d(v)$ is the number of edges incident with v. The maximum degree of the graph G is defined as $\Delta(G) = \max\{d(v)/v\epsilon V\}$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, the union $G_1 \cup G_2$ is defined to be $G = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, the sum $G_1 + G_2$ is defined as $G_1 \cup G_2$ together with all the lines joining points of V_1 to V_2. The Cartesian product of two graphs G_1 and G_2 denoted by $G = G_1 \times G_2$ is the graph G such that $V(G) = V(G_1) \times V(G_2)$, that is every vertex of $G_1 \times G_2$ is an ordered pair (u, v), where $u \in V(G_1)$ and $v \in V(G_2)$ and two distinct vertices (u, v) and (x, y) are adjacent in $G_1 \times G_2$ if either $u = x$ and $vy \in E(G_2)$ or $v = y$ and $ux \in E(G_1)$. The graph G with $V = S_1 \cup S_2 \cup \ldots \cup S_i \cup T$, where each S_i is a set of vertices having at least two vertices and having the same degree and $T = V \backslash \cup S_i$. The degree splitting graph of G denoted by $DS(G)$ and is obtained from G by adding the vertices $w_1, w_2, \ldots, w_t$ and joining w_i to each vertex of S_i, $1 \leq i \leq t$ [6]. For each vertex v of a graph G, take a new vertex v', join v' to all the vertices of G which are adjacent to v. The graph $s(G)$ thus obtained is called splitting graph of G [1]. The cycle consisting of n vertices is denoted by C_n. The graph $G = (V, E)$ is simply denoted by G.

2. Main Results:

Theorem: 2.1:

Let G be the (m, n)-Lollipop graph. Then the Vertex Polynomial of $V(G, x) = x^m + (m-1)x^{m-1} + (n-1)x^2 + x$.

Proof: Let G be the (m,n)-Lollipop graph and it has order $m+n$. In G, one vertex has degree m, $m-1$ vertices have degree $m-1$, $n-1$ vertices have degree 2 and one vertex has degree 1. This gives, $V(G,x) = x^m + (m-1)x^{m-1} + (n-1)x^2 + x$.

Example 2.2:

Let G be $(3,2)$-Lollipop graph. Then $V(G,x) = x^3 + 3x^2 + x$.

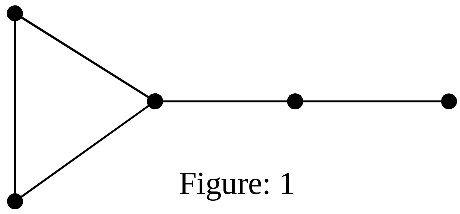

Figure: 1

Theorem 2.3:

Let G be the (m,n)-Lollipop graph. Then the Vertex Polynomial of $V(S(G),x) = x^{2m} + (m-1)x^{2(m-1)} + (n-1)x^4 + x^2 + x^m + (m-1)x^{m-1} + (n-1)x^2 + x$.

Proof: Let G be the (m,n)-Lollipop graph and it has order $m+n$. Then $S(G)$ has order $2(m+n)$, among this, one vertex has degree $2m$, $m-1$ vertices have degree $2(m-1)$, $n-1$ vertices have degree 4, one vertex has degree 2, one vertex has degree m, $m-1$ vertices have degree $m-1$, $n-1$ vertices have degree 2 and one vertex has degree 1. This gives, $V(S(G),x) = x^{2m} + (m-1)x^{2(m-1)} + (n-1)x^4 + x^2 + x^m + (m-1)x^{m-1}$
$+(n-1)x^2 + x$.

Theorem 2.4:

Let G be the (m,n)-Lollipop graph. Then the Vertex Polynomial of $V(DS(G),x) = x^{m+1} + (m-1)x^m + (n-1)x^3 + x^2 + x^{m-1} + x^{n-1} + 2x$.

Proof: Let G be the (m,n)-Lollipop graph and it has order $m+n$. Here, one vertex has degree m, $m-1$ vertices have same degree $m-1$, $n-1$ vertices have same degree 2 and one vertex has degree 1. Therefore in $DS(G)$, we can introduce four new vertices and make them adjacent with these $1, m-1, n-1$ and 1 vertex respectively. Hence, one vertex has degree 1, one vertex has degree $m-1$, one vertex has degree $n-1$, one vertex has degree one, one vertex has degree $m+1$, $m-1$ vertices have degree m, $n-1$ vertices have degree 3 and one vertex has degree 2. This gives, $V(DS(G),x) = x^{m+1} + (m-1)x^m + (n-1)x^3 + x^2 + x^{m-1} + x^{n-1} + 2x$.

Theorem: 2.5:

Let G be the (m,n)-Tadpole graph. Then the Vertex Polynomial of $V(G,x) = x^3 + (m-1)x^2 + (n-1)x^2 + x$.

Proof: Let G be the (m,n)-Tadpole graph and it has order $m+n$. In G, one vertex has degree 3, $m-1$ vertices have degree 2, $n-1$ vertices have degree 2 and one vertex has degree 1. This gives, $V(G,x) = x^3 + (m-1)x^2 + (n-1)x^2 + x$.

Example 2.6:

Let G be $(5,3)$-Tadpole graph. Then $V(G,x) = x^3 + 6x^2 + x$.

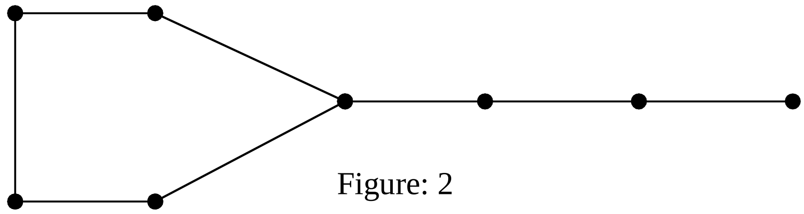

Figure: 2

Theorem 2.7:

Let G be the (m,n)-Tadpole graph. Then the Vertex Polynomial of $V(S(G),x) = x^6 + (m-1)x^4 + (n-1)x^4 + x^2 + x^3 + (m-1)x^2 + (n-1)x^2 + x$.

Proof: Let G be the (m,n)-Tadpole graph and it has order $m+n$. Then $S(G)$ has order $2(m+n)$, among this, one vertex has degree 6, $m-1$ vertices have degree 4, $n-1$ vertices have degree 4, one vertex has degree 2, one vertex has degree 3, $m-1$ vertices have degree 2, $n-1$ vertices have degree 2 and one vertex has degree 1. This gives, $V(S(G),x) = x^6 + (m-1)x^4 + (n-1)x^4 + x^2 + x^3 + (m-1)x^2 + (n-1)x^2 + x$.

Theorem 2.8:

Let G be the (m,n)-Tadpole graph. Then the Vertex Polynomial of $V(DS(G),x) = x^4 + (m-1)x^3 + (n-1)x^3 + x^2 + x^{m-1} + x^{n-1} + 2x$.

Proof: Let G be the (m,n)-Tadpole graph and it has order $m+n$. Here, one vertex has degree 3, $m-1$ vertices have same degree 2, $n-1$ vertices have same degree 2 and one vertex has degree 1. Therefore in $DS(G)$, we can introduce four new vertices and make them adjacent with these $1, m-1, n-1$ and 1 vertex respectively. Hence, one vertex has degree 1, one vertex has degree $m-1$, one vertex has degree $n-1$, one vertex has degree one, one vertex has degree 4, $m-1$ vertices have degree 3, $n-1$ vertices have degree 3 and one vertex has degree 2. This gives, $V(DS(G),x) = x^4 + (m-1)x^3 + (n-1)x^3 + x^2 + x^{m-1} + x^{n-1} + 2x$.

Theorem 2.9:

Let C_m and C_n be Cycles with order m and n respectively. Then $S(C_m \cup C_n, x) = 2(m+n)x^{4(m+n)}$.

Proof: Let C_m and C_n be Cycles with order m and n respectively. In C_m, m vertices have degree 2. In C_n, n vertices have degree 2. Therefore in this graph $C_m \cup C_n$, mn vertices have degree 2. Hence, sum of the number of adjacent vertices of all the vertices of degree 2 is 2mn, sum of the degree of adjacent vertices of all the vertices of degree 2 is 4mn. This gives, $S(C_m \cup C_n, x) = 2(m+n)x^{4(m+n)}$.

Example 2.10:

Consider the graph $C_3 \cup C_4$, then $S(C_3 \cup C_4, x) = 14x^{28}$.

$C_3 \cup C_4$:

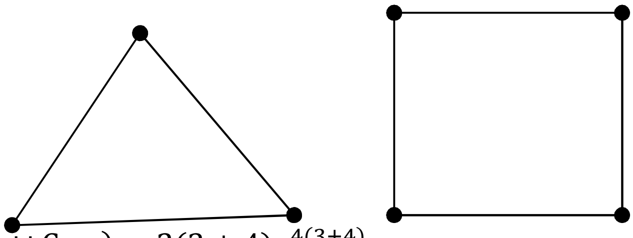

Figure: 3

Here, $S(C_3 \cup C_4, x) = 2(3+4)x^{4(3+4)}$.

$S(C_3 \cup C_4, x) = 14x^{28}$.

Theorem 2.11:

Let C_m and C_n be Cycles with order m and n respectively. Then $S(C_m + C_n, x) = m(n+2)x^{2(n+2)+n(m+2)} + n(m+2)x^{2(m+2)+m(n+2)}$.

Proof: Let C_m and C_n be Cycles with order m and n respectively. In C_m, m vertices have degree 2. In C_n, n vertices have degree 2. Therefore in this graph $C_m + C_n$, m vertices have degree $n+2$, n vertices have degree $m+2$. Hence, sum of the number of adjacent vertices of all the vertices of degree $n+2$ is $m(n+2)$, sum of the degree of adjacent vertices of all the vertices of degree $n+2$ is $2(n+2)+n(m+2)$, sum of the number of adjacent vertices of all the vertices of degree $m+2$ is $n(m+2)$, sum of the degree of adjacent vertices of all the vertices of degree $m+2$ is $2(m+2)+m(n+2)$. This gives, $S(C_m + C_n, x) = m(n+2)x^{2(n+2)+n(m+2)} + n(m+2)x^{2(m+2)+m(n+2)}$.

Example 2.12:

Consider the graph $C_3 + C_4$, then $S(C_3 + C_4, x) = 14x^{28}$.

$C_3 + C_4$:

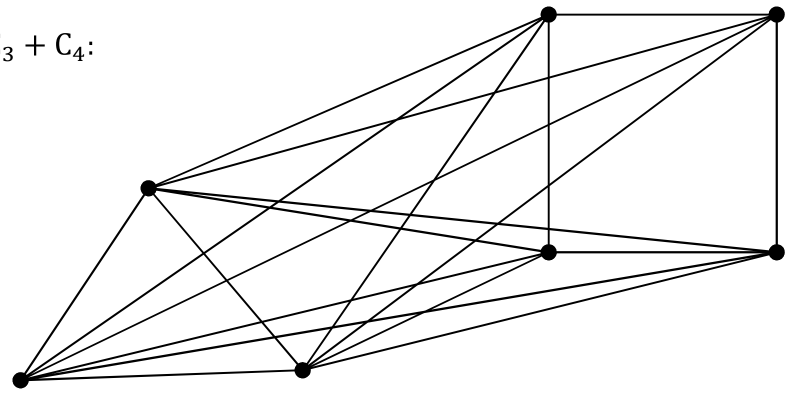

Figure: 4

Here, $S(C_3 + C_4, x) = 3(4+2)x^{2(4+2)+4(3+2)} + 4(3+2)x^{2(3+2)+3(4+2)}$.

$S(C_3 + C_4, x) = 18x^{32} + 20x^{28}$.

Theorem 2.13:

Let C_m and C_n be Cycles with order m and n respectively. Then $S(s(C_m UC_n), x) = 4(m+n)x^{12(m+n)} + 2(m+n)x^{8(m+n)}$.

Proof: Let C_m and C_n be Cycles with order m and n respectively. In C_m, m vertices have degree 2. In C_n, n vertices have degree 2. Therefore in this graph $s(C_m UC_n)$, m and n vertices have degree 4, m *and* n vertices have degree 2. Hence, sum of the number of adjacent vertices of all the vertices of degree 4 is $4(m+n)$, sum of the degree of adjacent vertices of all the vertices of degree 4 is $12(m+n)$, sum of the number of adjacent vertices of all the vertices of degree 2 is $2(m+n)$, sum of the degree of adjacent vertices of all the vertices of degree 2 is $8(m+n)$. This gives, $S(s(C_m UC_n), x) = 4(m+n)x^{12(m+n)} + 2(m+n)x^{8(m+n)}$.

Example 2.14:

Consider the graph $s(C_4 \cup C_4)$, then $S(s(C_4 \cup C_4), x) = 32x^{96} + 16x^{64}$.

$s(C_4 \cup C_4)$:

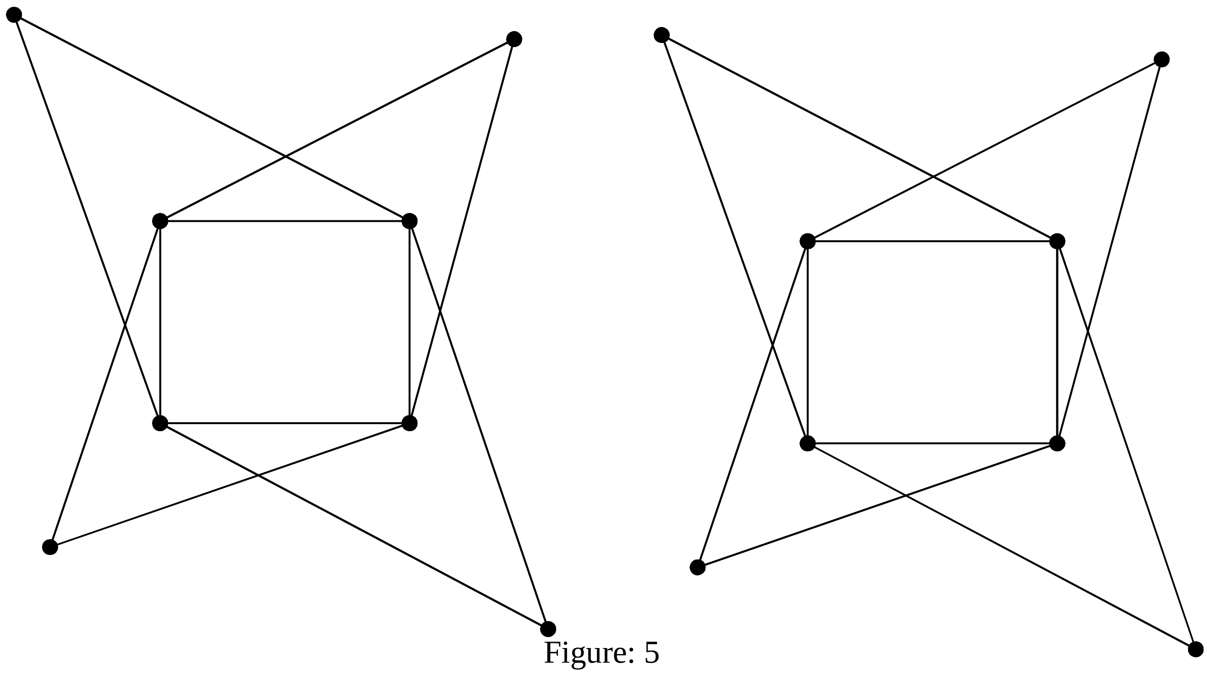

Figure: 5

Then $S(s(C_4 UC_4), x) = 4(4+4)x^{12(4+4)} + 2(4+4)x^{8(4+4)}$.

$S(s(C_4 UC_4), x) = 32x^{96} + 16x^{64}$.

Theorem 2.15:

Let C_m and C_n be Cycles with order m and n respectively. Then $S(DS(C_m UC_n), x) = 3(m+n)x^{m(m+6)+n(n+6)} + mx^{3m} + nx^{3n}$.

Proof: Let C_m and C_n be Cycles with order m and n respectively. In C_m, m vertices have degree 2. In C_n, n vertices have degree 2. Therefore in this graph $DS(C_m UC_n)$, m and n vertices have degree 3, 1 vertex has degree m and 1 vertex has degree n. Hence, sum of the number of adjacent vertices of all the vertices of degree 3 is

$3(m+n)$, sum of the degree of adjacent vertices of all the vertices of degree 3 is $m(m+6)+n(n+6)$, sum of the number of adjacent vertices of all the vertices of degree m is m, sum of the degree of adjacent vertices of all the vertices of degree m is $3m$, sum of the number of adjacent vertices of all the vertices of degree n is n, sum of the degree of adjacent vertices of all the vertices of degree n is $3n$. This gives, $S(DS(C_m UC_n), x) = 3(m+n)x^{m(m+6)+n(n+6)} + mx^{3m} + nx^{3n}$.

Example 2.16:

Consider the graph $DS(C_4 \cup C_4)$, then $S(DS(C_4 \cup C_4), x) = 24x^{80} + 4x^{12} + 4x^{12}$.

$DS(C_4 \cup C_4)$:

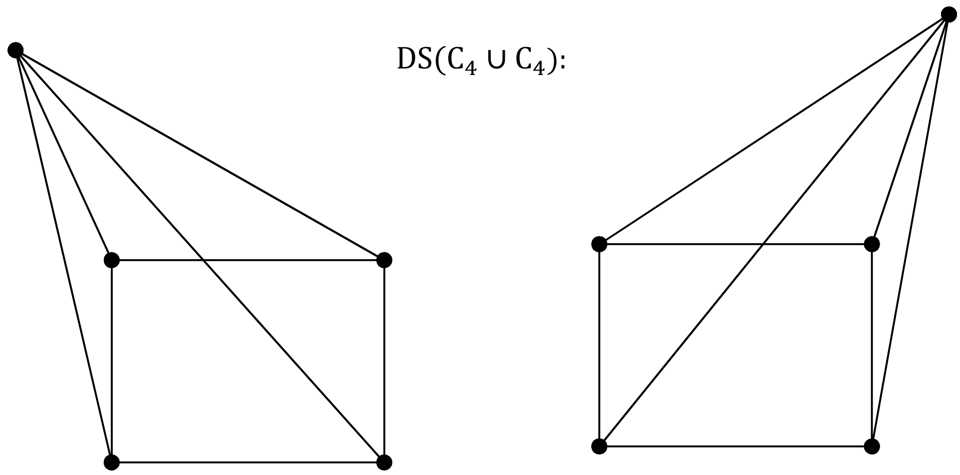

Figure: 6

Here, $S(DS(C_4 UC_4), x) = 3(4+4)x^{4(4+6)+4(4+6)} + 4x^{12} + 2x^{12}$.

$S(DS(C_4 UC_4), x) = 24x^{80} + 4x^{12} + 2x^{12}$.

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