Derivation of Newton's law of motion from Kepler's law's of planetary motion

Rajni
Department of Mathematics, S.K. Govt. College, Kanwali, Rewari, Haryana, India

Abstract

In the first quarter of the 17^{th} century Johannes Kepler (1571-1630) has formulated his three laws regarding the movement of planets. Kepler's has deduced these laws from measured data. The law he has formulated are valid for every space object flying in a gravitational field, for example satellites, scientist probes or capsules moving around the earth or other planet.

Newton's law of motion is derived from Kepler's law's of planetary motion. This is achieved by applying a simple system identification method using numerical data from the planet's orbits in conjunction with the inverse square law's for the attractive force between celestial bodies and the concepts of the derivative and differential equation. The identification procedure fields the differential equation of motion of a body under the action of an applied force as stated by Newton. Moreover, besides Validating the inverse square law, Newton's law of universal gravitation. As the employed mathematical tools and the theory were available before 1686, we are allowed to state that the equation of motion for a body with constant mass could have been established from kepler's law of planetary motion, before Newton had published his law of motion.

KEYWORD - Newton's law of motion, Kepler's laws planetary motion, Inverse square law system identification.

Introduction

First , we deduce the three kepler's laws from Newton's laws of ravitation.

(1) If the central orbit is an ellipse under a force towards one of itsfoci , then the attraction of the particle varies inversely as the square of its distance from the focus. Hence from the first Kepler's law we conclude that the acceleration of each planet toward the sun varies inversely as the square of its distance from the sun.

(ii) According to Kepler's second law, the radius vector drawn from the sun to a planet sweeps out equal areas in equal times. Therefore the rate of description of the sectorial area is constant which is true only in central orbits. Hence from this law we conclude that the acceleration of the planet is directed towards the sun.

(iii) According to Kepler's third law

$T^2 \propto (2a)^3$, where 2a is major axis of the orbit.

In central orbits, we have

$$T = \frac{\textbf{\textit{Area of ellipse}}}{\textbf{\textit{Rate of description of the sectorial area}}}$$

$$= \frac{\pi ab}{\frac{1}{2}h} = \frac{2\pi ab}{h}$$

$$= \frac{2\pi ab}{\sqrt{\mu l}}$$, Where l is the semi latus - rectum

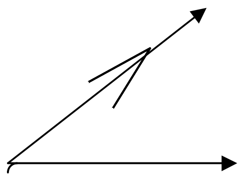
$$= \frac{2\pi ab}{\sqrt{\mu \frac{b^2}{a}}} = \frac{2\pi a^{3/2}}{\sqrt{\mu}} \left[l = \frac{b^2}{a}\right]$$

$$T^2 = \frac{4\pi^2 a^3}{\mu} = \frac{\pi^2 (2a)^3}{2\mu}$$

From Kepler's third law, $T^2 \propto (2a)^3$. This shows that the absolute acceleration μ is same for all planets.

Take the sun as fixed particle in space and its centre as the orgin. Let P(r,$\boldsymbol{\theta}$) be the position of planet at time t. Then by Newton's law of gravitation. The force F per unit mass of the planet is given by

$F = \frac{\mu}{r^2}$, *Where* μ *is a positive constant* P(r, $\boldsymbol{\theta}$)



The differential equation of the central orbit is $\frac{\mu}{r^2}$

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}\theta$$

or $\frac{d^2u}{d\theta^2} + u = \frac{\mu u^2}{h^2 u^2}$ Where $U = \frac{1}{r}$o x

or $\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2} = 0$

or $\frac{d^2}{d\theta^2}\left(u - \frac{\mu}{h^2}\right) + \left(u - \frac{\mu}{h^2}\right) = 0$

or $\frac{d^2x}{dt^2} + x = 0$ Where $x = u - \frac{\mu}{h^2}$

or $(D^2 + 1)\, x = 0$ -----------------------(1)

or Auxiliary equation is $D^2 + 1 = 0$ or $D^2 = -1$ i.e., $D = \pm i$

Thus the general solution of (1) is

$\boldsymbol{x} = c_1 \cos(\boldsymbol{\theta} - c_2)$ where c_1 and c_2 are arbitrary constants

Putting $\boldsymbol{x} = u - \frac{\mu}{h^2}$, ***we get***

$u - \frac{\mu}{h^2} = c_1 \cos(\boldsymbol{\theta} - c_2)$ $\left[u = \frac{1}{r}\right]$

or $\frac{1}{r} - \frac{\mu}{h^2} = \mathbf{C_1 Cos}\,(\boldsymbol{\theta} - \mathbf{C2})$

Dividing both sides by $\frac{\mu}{h^2}$, we get

$$\frac{\frac{h2}{\mu}}{r} - 1 = \frac{c_{1}h^2}{\mu}\cos\ (\boldsymbol{\theta} - c_2)$$

or $$\frac{\frac{h2}{\mu}}{r} = 1 + \frac{c_{1}h^2}{\mu}\cos\ (\boldsymbol{\theta} - c_2)$$

Which is polar equation of a conic of the form $\frac{1}{r}$ = 1 + e cos$\boldsymbol{\theta}$with the sun at one focus, The semi lotus - Rectum is $\frac{h^2}{\mu}$and eccentricity is $\frac{c_{1}h^2}{\mu}$. Since ellipse is the only closed conic, there for the planet describes an ellipse with sun at one focus. This is Kepler's first law since the rate of change of the sectorial area traced out by the radius vector joining the particle to a fixed point is constant and equal to $\frac{h}{2}$ for the central orbits.

Therefore Kepler's second law is verified.

Further let a and b be the lengths of the semi-major and semi-minor axes of the elliptic orbit respectively . If T is the time of one revolution along the orbit, then

$$T = \frac{\textit{Area of the ellipse}}{\textit{Rate of description of the sectorial area}}$$

Now we shall assume that three Kepler's laws holds good and deduce the Newton's law of gravitation.

From Kepler's second law, the areal velocity is constant i.e. $\frac{h}{2}$ is constant.

Therefore, $\frac{1}{2}(r^2\frac{d}{dt}\boldsymbol{\theta})$ = constant

$$\frac{d}{dt} \Rightarrow (r^2\frac{d}{dt}\boldsymbol{\theta}) = 0 \frac{1}{r}\frac{d}{dt}(r^2\frac{d}{dt}\boldsymbol{\theta}) = 0 \Rightarrow$$

Thus the transverse component of the acceleration of the particle vanishes and the planet has only radial acceleration directed towards the sun, which implies that the force of attraction is central and directed towards the sun.

From Kepler's first law, the path of a planet is elliptic about the sun. The equation of the ellipse having pole at the sun as focus is

$$\frac{l}{r} = 1 + e\cos\boldsymbol{\theta}\text{ , where } e < 1$$

or $$ul = 1 + e\cos\boldsymbol{\theta} \text{[therefore, } u = \frac{1}{r}\text{]}$$

or $$u = \frac{1}{l}[\,1 + e\cos\boldsymbol{\theta}\,]$$

Differentiating w.rt□, we have

$\frac{du}{d\theta} = \frac{1}{l}[\, - e \sin \theta \,]$ and $\frac{d^2u}{d\theta^2} = - \frac{e}{l} \cos \theta$ The force F per unit mass for the central orbit is given by

$$F = h^2 u^2 \left[u + \frac{d^2u}{d\theta^2} \right]$$

$$== h^2 u^2 \left[\frac{1+e\cos\theta}{l} - \frac{e\cos\theta}{l} \right]$$

$$= h^2 u^2 \left[\frac{1}{l}\right] = \frac{h^2}{r^2}\left[\frac{1}{l}\right]$$ T = Time period for an elliptic orbit

$$= \frac{\text{Area of the ellipse}}{\text{Rate of description of the sectorial area}}$$

$$= \frac{\pi ab}{\frac{h}{2}} \text{------------}(2)$$

According Kepler's third law,

$$T^2 \propto (2a)^3$$

$$\frac{T^2}{a^3} = \text{Constant}$$

i.e. $\frac{4\pi^2 a^2 b^2}{h^2 a^3} = \text{Constant}$

or $\frac{4\pi^2}{h^2}\left(\frac{b^2}{a}\right) = \text{Constant}$

or $\frac{4\pi^2}{h^2}(\ell) = \text{Constant}$ $\left[\frac{b^2}{a} = \ell\right.$

or $\frac{l}{h^2}$ = Constant for all planets

or $\frac{h^2}{l} = \text{Constant} = \mu$

from (2), $F = \frac{1}{r^2}\left(\frac{h^2}{l}\right) = \frac{1}{r^2}(\mu)$

therefore, $mF = \frac{m\mu}{r^2}$

Thus the force of attraction between the sun and the planet of mass m varies inversely as the square of the distance between them . This shows that Newton's law of gravitation follows from Kepler's laws of planetary motion.

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